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GENERALIZED ASSOCIATIVITY ON GROUPOIDS

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The functional equation of associativity, either in its general form or in some special case, has been studied by many mathematicians: Suškevič, Aczél, Belousov, Hosszú, Schauffler, Devidé, Prešić, Milić and other.

The most striking of all results about associativity equation is probably the Four quasigroups theorem ([1]).

According to this Theorem, quasigroups satisfying generalized associativity equation are all istopes of the same group.

Here we give the general solution of generalized associativity equation without any assumptions about functions involved.

As consequences, associativity criterion for (finite) groupoids, reducibility criterion for ternary operations, analogue of Schauffler theorem and other example are given.

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Definition 1. We say that ternary operation T on S is reducible iff there are binary operations A and B such that either

$$T(x_1, x_2, x_3) = A(x_i, B(x_j, x_k)) \text{ or } T(x_1, x_2, x_3) = A(B(x_i, x_j)x_k)$$

where i, j, k, are from $\{1, 2, 3\}$ and pairwise different.

Definition 2. Left, middle and right translation of ternary operation T on S are defined by:

$$\lambda_{xy}z = \mu_{xz}y = \rho_{yz}x = T(x, y, z).$$

The set of all functions from S to S we denote by τ_s.

Also:

$$\tau_1 = \{\rho_{yz} \mid y \in S,\ \ z \in S\}$$
$$\tau_2 = \{\mu_{xz} \mid x \in S,\ \ z \in S\}$$
$$\tau_3 = \{\lambda_{xy} \mid x \in S,\ \ y \in S\}.$$

Definiton 3. Let α be an equivalence on S^2. Then

$$(x,y,z)\alpha_1(u,v,w) \text{ iff } x = u \text{ and } (y,z)\alpha(v,w)$$
$$(x,y,z)\alpha_2(u,v,w) \text{ iff } y = y \text{ and } (z,x)\alpha(w,u)$$
$$(x,y,z)\alpha_3(u,v,w) \text{ iff } (x,y)\alpha(u,v) \text{ and } z = w$$

Definition 4. Let α, β, γ be equivalences on S^2. Then

$$\alpha \,\square\, \gamma = \alpha_1 \vee \gamma_3$$
$$E(\alpha,\beta,\gamma) = \alpha_1 \vee \beta_2 \vee \gamma_3$$

It is easy to see that all relations α_1, $\alpha_2(\beta_2)$, $\alpha_3(\gamma_3)$, $\alpha \,\square\, \gamma$ and $E(\alpha,\beta,\gamma)$ are equivalences on S^3. By $\vee$ we denote the supremum operation in lattice of all equivalences on some set.

Theorem 5. *The general solution (on a nonempty set S) of the generalized associativity equation*

$$A(x, B(y,z)) = C(D(x,y),z) \tag{1}$$

is given by:

$$\begin{aligned} A(x,y) &= (fy)x \\ B(x,y) &= P(x,y) \\ C(x,y) &= (gx)y \\ D(x,y) &= Q(x,y) \end{aligned} \tag{2}$$

where:

(i) *P and Q are arbitrary groupoids on S*

(ii) *$f: S \to \tau_s$ and $g: S \to \tau_s$ are arbitrary functions such that:*

$$fP(x,y) = \rho_{xy} \quad gQ(x,y) = \lambda_{xy} \tag{3}$$

where $\lambda_{xy}(\rho_{xy})$ is left (right) translation of an arbitrary ternary operation T on S satisfying:

$$\ker P \,\square\, \ker Q \subset \ker T. \tag{4}$$

PROOF: (a) For given P, Q, T, f and g (satisfying (i) and (ii)) quadruple (A,B,C,D) given by (2) is a solution of (1), as it can be easily checked. Existence of f and g satisfying (ii) is quarantied by (4).

(b) Let A,B,C,D be groupoids satisfying (1) and let $P(x,y) = B(x,y)$, $Q(x,y) = D(x,y)$, $T(x,y,z) = A(x,B(y,z))$, $(fx)y = A(y,x)$ if $x \in P(S,S)$ and arbitrary otherwise, $(gx)y = C(x,y)$ if $x \in Q(S,S)$ and arbitrary otherwise.

From $P(x,y) = P(u,y)$ it follows that:

$$B(x,y) = B(u,v)$$
$$A(z,B(x,y)) = A(z,B(u,v))$$
$$T(z,x,y) = T(z,u,v)$$

so $(z,x,y) \ker T(z,u,v)$ for all $z \in S$.

Analogously from $Q(x,y) = Q(u,v)$ it follows that $(x,y,z) \ker T(u,v,z)$ for all $z \in S$. Using D 4, (4) then easily follows.

From the definition of f we have:

$$(fP(x,y))z = A(z,B(x,y)) = T(z,x,y) = \rho_{xy}z$$

so $fP(x,y) = \rho_{xy}$. Anologously $gQ(x,y) = \lambda_{xy}$.

The following theorem gives a reducibility criterion for ternary operations.

THEOREM 6. *Ternary operation T (on S) is reducible iff $|\tau_1| \leq |S|$ or $|\tau_2| \leq |S|$ or $|\tau_3| \leq |S|$.*

PROOF. (a) If T is reducible then for example $T(x,y,z) = A(x,B(y,z))$ for some binary operations A and B. It follows that

$$B(y,z) = B(u,v) \Rightarrow \rho_{xy} = \rho_{uv} \text{ and } |\tau_1| \leq |B(S,S)| \leq |S|.$$

If T is expresible in some other way, analogous procedure shows that at least one of τ_1, τ_2, τ_3 has no more than $|S|$ elements.

(b) Let some of τ_1, τ_2, τ_3, for example τ_1, has no more elements than S. Then groupoid B can be defined so that:

$$\rho_{xy} \neq \rho_{uv} \Rightarrow B(x,y) \neq B(u,v) \tag{5}$$

Let $f_0 \colon B(S,S) \to \tau_1$ be defined by $f_0 B(x,y) = \rho_{xy}$. From (5) it follows that f_0 is well defined and surjection. Let A be an arbitrary groupoid such that $A(x,y) = (f_0 y)x$ for all $y \in B(S,S)$. Then $T(x,y,z) = \rho_{yz}x = (f_0 B(y,z))x = A(x,B(y,z))$ so T is reducible.

Analogously if $|\tau_2| \leq |S|$ or $|\tau_3| \leq |S|$.

THEOREM 7. *Let A be a binary operation and let $T(x,y,z) = A(x,B(y,z))$. Then, A is associative iff $g_0 = \{(A(x,y),\ \lambda_{xy}) \mid x,y \in S\}$ is a function from $A(S,S)$ to τ_3 and also $(g_0 x)y = A(x,y)$ for all $x \in A(S,S)$.*

PROOF: (a) If A is associative then g_0 defined by $(g_0x)y = A(x,y)$ for $x \in A(S,S)$, is a function, as follows from the proof of Th 5 (b). Also $(g_0A(u,v))y = A(A(u,v),y) = A(u,A(v,y)) = T(u,v,y) = \lambda_{uv}y$.

(b) Let $g_0\colon A(x,y) \mapsto \lambda_{xy}$ be a function from $A(S,S)$ to τ_3 such that $(g_0x)y = A(x,y)$ for all $x \in A(S,S)$ and let $P(x,y) = Q(x,y) = A(x,y)$.

In order to prove (ii) of Th 5 let us suppose that (x,y,z) $\ker P \,\square\, \ker Q$ (u,v,w). The there is a sequence $(p_i,q_i,r_i)_{i=1,\dots,n}$ such that:

$$\begin{aligned}&(x,y,z) = (p_1,q_1,r_1), \text{ either } (p_i,q_i,r_i)(\ker P)_1(p_{i+1},q_{i+1},r_{i+1}) \text{ or}\\ &(p_i,q_i,r_i)(\ker Q)_3(p_{i+1},q_{i+1},r_{i+1}) \text{ for } i=1,\dots,n-1 \text{ and}\\ &(p_n,q_n,r_n) = (u,v,w).\end{aligned}$$

From $(p_i,q_i,r_i)(\ker P)_1(p_{i+1},q_{i+1},r_{i+1})$ it follows that

$$\begin{aligned}&p_i = p_{i+1},\ \ P(q_i,r_i) = P(q_{i+1},r_{i+1})\\ &p_i = p_{i+1},\ \ A(q_i,r_i) = A(q_{i+1},r_{i+1})\\ &A(p_i,A(q_i,r_i)) = A(p_{i+1},\ \ A(q_{i+1},r_{i+1}))\\ &T(p_i,q_i,r_i) = T(p_{i+1},q_{i+1},r_{i+1}).\end{aligned}$$

From $(p_i,q_i,r_i)(\ker Q)_3(p_{i+1},q_{i+1},r_{i+1})$ it follows that

$$\begin{aligned}&Q(p_i,q_i) = Q(p_{i+1},q_{i+1}),\ \ r_i = r_{i+1}\\ &A(p_i,q_i) = A(p_{i+1},q_{i+1}),\ \ r_i = r_{i+1}\\ &g_0A(p_i,q_i) = g_0A(p_{i+1},q_{i+1}),\ \ r_i = r_{i+1}\\ &\lambda_{p_iq_i} = \lambda_{p_{i+1}q_{i+1}},\ \ r_i = r_{i+1}\\ &\lambda_{p_iq_i}r_i = \lambda_{p_{i+1}q_{i+1}}r_{i+1}\\ &T(p_i,q_i,r_i) = T(p_{i+1},q_{i+1},r_{i+1}).\end{aligned}$$

In any case (p_i,q_i,r_i) $\ker T$ $(p_{i+1},q_{i+1},r_{i+1})$ for all $i=1,\dots,n-1$. Transitivity of $\ker T$ ensures $(x,y,z)\ker T(u,v,w)$ as should be proved.

Let f and g be given with $(fx)y = A(y,x)$ and

$$(gx)y = \begin{cases}(g_0x)y, & x \in A(S,S)\\ A(x,y), & x \in S\backslash A(S,S)\end{cases}.$$ Then $A(x,y) = (fy)x$, $A(x,y) = P(x,y)$,

$$\begin{aligned}&A(x,y) = (gx)y,\ \ A(x,y) = Q(x,y) \text{ and}\\ &(fP(x,y))z = (fA(x,y))z = A(z,\ \ A(x,y)) = T(z,x,y) = \rho_{xy}z\\ &(gQ(x,y))z = (gA(x,y))z = (g_0A(x,y))z = \lambda_{xy}z\end{aligned}$$

so (A,A,A,A) is a solution of (1) i.e. A is associative.

Using Th 7 we can easily chech if a finite groupoid is associative.

– For all $y \in S$ form Cayley table of $T_y(T_y(x,z) = T(x,y,z))$

– check if

$$A(x,y) = A(u,v) \Rightarrow \lambda_{xy} = \lambda_{uv} \tag{6}$$

hold

– if (6) does not hold for some $x,y,u,v \in S$ then A cannot be associative

– if (6) holds, define $g_0\colon A(s,s) \to \tau_3$ by $g_0 A(x,y) = \lambda_{xy}$

– define partial groupoid C_0 by:

$$C_0(x,y) = (g_0 x)y \quad (x \in A(S,S))$$

– if Cayley table for C_0 is a part of Cayley table for A then A is associative.

This associativity criterion differs from well known Light's test ([2]).

Example 1. Let unknown groupoids from (1) be quasigroups. According to [3], the general solution of (1), in this case, can be obtained in the following from:

$$\begin{aligned} A(x,y) &= A_1 x \cdot A_2 y \\ B(x,y) &= A_2^{-1}(A_2 B_1 x \cdot A_1 B_2 y) \\ C(x,y) &= C_1 x \cdot C_2 y \\ D(x,y) &= C_1^{-1}(C_1 D_1 x \cdot C_1 D_2 y) \end{aligned} \tag{7}$$

where $\cdot$ is an arbitrary group and $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are arbitrary permutations such that:

$$\begin{aligned} A_1 &= C_1 D_1 \\ A_2 B_1 &= C_1 D_2 \\ A_2 B_2 &= C_2 \end{aligned} \tag{8}$$

Our aim is to prove that this solution we can obtain from the general solution in the groupoid case.

Let $p,q,r \in S$, $b = B(b,r)$, $d = D(p,q)$ and $e = A(p,b)$. Let also:

$$\begin{aligned} A_1 x &= (fb)x & A_2 x &= (fx)p \\ B_1 x &= P(x,r) & B_2 x &= P(q,x) \\ C_1 x &= (gx)r & C_2 x &= (gd)x \\ D_1 x &= Q(x,q) & D_2 x &= Q(p,x) \end{aligned} \tag{9}$$

Then

$$\begin{aligned}
A_1x &= (fb)x = fB(q,r))x = (fP(q,r))x = \rho_{qr}x = \\
&= T(x,q,r) = \lambda_{xq}r = (gQ(x,q))r = C_1Q(x,q) = C_1D_1x \\
A_2B_1x &= A_2P(x,r) = (fP(x,r))p = \rho_{xr}p = T(p,x,r) = \\
&= \lambda_{px}r = (gQ(p,x))r = C_1Q(p,x) = C_1D_2x \\
A_2B_2x &= A_2P(q,x) = (fP(q,x)p = \rho_{qx}p = T(p,q,x) = \\
&= \lambda_{pq}x = (gQ(p,q))x = (gD(p,q))x = (gd)x = C_2x
\end{aligned}$$

so (8) holds.

Since A, B, C, D are quasigroups $A_1, A_2, B_1, B_2, C_1, C_2, D_1$ and D_2 defined by (9) are permutations.

Also:

$$\begin{aligned}
A(x,B_1y) &= (fB_1y)x = (fP(y,r))x = \rho_{yr}x = T(x,y,r) = \\
&= \lambda_{xy}r = (gQ(x,y))r = C_1Q(x,y) = C_1D(x,y) \\
A_2B(x,y) &= (fB(x,y))p = (fP(x,y))p = \rho_{xy}p = T(p,x,y) = \\
&= \lambda_{px}y = (gQ(p,x))y = (gD_2x)y = C(D_2x,y) \\
(fB_2y)x &= (fP(q,y))x = \rho_{qy}x = T(x,q,y) = \\
&= \lambda_{xq}y = (gQ(x,q))y = (gD_1x)y
\end{aligned}$$

Let $x \cdot y = (fA_2^{-1}y)A_1^{-1}x$. We get

$$\begin{aligned}
A(x,y) &= (fy)x = (fA_2^{-1}A_2y)A_1^{-1}A_1x = A_1x \cdot A_2y \\
C(x,y) &= (gx)y = (gD_1D_1^{-1}x)y = (fB_2y)D_1^{-1}x = \\
&= (fA_2^{-1}A_2B_2y)A_1^{-1}A_1D_1^{-1}x = A_1D_1^{-1}x \cdot A_2B_2y = \\
&= C_1D_1D_1^{-1}x \cdot C_2y = C_1x \cdot C_2y \\
B(x,y) &= A_2^{-1}C(D_2x,y) = A_2^{-1}(C_1D_2x \cdot C_2y) = A_2^{-1}(A_2B_1x \cdot A_2B_2x) \\
D(x,y) &= C_1^{-1}A(x,B_1y) = C_1^{-1}(A_1x \cdot A_2B_1y) = C_1^{-1}(C_1D_1x \cdot C_1D_2y)
\end{aligned}$$

so we obtained (7).

Associativity of $\cdot$ also easily follows.

EXAMPLE 2. In [4] Schauffler proved the following theorem:

THEOREM 2.1. *For any two quasigroups A, B on S there are quasigroups C, D (on S) such that* (1) *holds iff* $|S| \leq 3$.

Here is an analogous theorem for groupoids.

THEOREM 2.2. *For any two groupoids A, B on S there are quasigroups C, D (on S) such that* (1) *holds iff S is infinite or* $|S| = 1$.

PROOF: Let $T(x,y,z) = A(x, B(y,z))$.

(a) If S is infinite or $|S| = 1$ then $|\tau_3| \leq |S^2| = |S|$ and, according to Th 6, $T(x,y,z) = C(D(x,y),z)$ for some groupoids C, D on S.

(b) Let $n > 1$, $S = \{a_1, \ldots, a_n\}$ and:

$$\begin{aligned}
&M_i x = x \text{ for } i = 1, 3, \ldots, n\\
&M_2 x = x \text{ for } x \neq a_1, a_2\\
&M_2 a_1 = a_2\\
&M_2 a_2 = a_1\\
&N_i a_1 = a_i \text{ for } i = 1, \ldots, n\\
&N_i x = x \text{ for } x \neq a_1 \text{ and } i = 1, \ldots, n\\
&A(a_i, a_j) = M_i a_j \text{ for } i, j = 1, \ldots, n\\
&B(a_i, a_j) = N_i a_j \text{ for } i, j = 1, \ldots, n
\end{aligned}$$

Since no two of $N_i (i = 1, \ldots, n)$ are equal and $\lambda_{a_1 a_i} = M_1 N_i = N_i$, $|\tau_3| \geq n$. Also, $\lambda_{a_2 a_1} = M_2 N_1 = M_2 \neq N_i$ for all $i = 1, \ldots, n$ and consequently $|\tau_3| > n$.

According to Th 6 we cannot express T in the form $C(D(x,y),z)$ whatever groupoids C, D we use.

COROLLARY 2.3. *Let groupoids A, B on S be given. Functional equation*

$$C(D(x,y),z) = A(x, B(y,z)) \tag{10}$$

has a solution iff $|\tau_3| \leq |S|$ where $T(x,y,x) = A(x, B(y,z))$.

If (10) *can be solved, its general solution is given by:*

$$\begin{aligned}
C(x,y) &= (gx)y\\
D(x,y) &= Q(x,y)
\end{aligned} \tag{11}$$

where Q is an arbitrary groupoid such that

$$\lambda_{xy} \neq \lambda_{uv} \Rightarrow Q(x,y) \neq Q(u,v)$$

and $g: S \to \tau_s$ an arbitrary function such that

$$gQ(x,y) = \lambda_{xy}.$$

COROLLARY 2.4. *Any ternary operation on an infinite set is reducible.*

EXAMPLE 3. (Generalized cyclic associativity)

Theorem 3.1. *Let the following system of functional equations be given:*

$$A(x,B(y,z)) = C(y,D(z,x)) = E(z,F(x,y)) \tag{12}$$

The general solution of (12) *is given by:*

$$\begin{aligned} A(x,y) &= (fy)x \\ B(x,y) &= P(x,y) \\ C(x,y) &= (gy)x \\ D(x,y) &= Q(x,y) \\ E(x,y) &= (hy)x \\ F(x,y) &= R(x,y) \end{aligned} \tag{13}$$

where:

(i) P, Q, R *are arbitrary groupoids on* S

(ii) f, g, h *are arbitrary functions from* S *to* τ_s *such that*

$$fP(x,y) = \rho_{xy} \quad gQ(x,y) = \mu_{yx} \quad hR(x,y) = \lambda_{xy} \tag{14}$$

where $\lambda_{xy}(\mu_{yx},\rho_{xy})$ *is left (middle, right) translation of an arbitrary ternary operation* T *on* S *satisfying:*

$$E(\ker P,\ \ker Q,\ \ker R) \subset \ker T \tag{15}$$

The proof is similar to the proof of Th 5.

Corollary 3.2. *The general solution of* (12)*, in the case where* A, B, C, D, E, F *are quasigroups, is given by:*

$$\begin{aligned} A(x,y) &= A_1x \cdot A_2y \\ B(x,y) &= A_2^{-1}(A_2B_1x \cdot A_2B_2y) \\ C(x,y) &= C_2y \cdot C_1x \\ D(x,y) &= C_2^{-1}(C_2D_2y \cdot C_2D_1x) \\ E(x,y) &= E_2y \cdot E_1x \\ F(x,y) &= E_2^{-1}(E_2F_1x \cdot E_2F_2y) \end{aligned} \tag{16}$$

where $\cdot$ *is an arbitrary abelian group on* S *and* A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2 *arbitrary permutations such that:*

$$\begin{aligned} A_1 &= C_2D_2 = E_2F_1 \\ A_2B_1 &= C_1 = E_2F_2 \\ A_2B_2 &= C_2D_1 = E_1. \end{aligned}$$

The proof is similar to the proof in example 1, using Th 3.1 instead of Th 5. See [3, II] example 2.

EXAMPLE 4. Let the unknown function from (1) be G-groupoids. According to [5], G-groupoid is a function from $S_1 \times S_2$ to S. For the sake of definiteness let:

$$A: S_1 \times S_4 \to S$$
$$B: S_2 \times S_3 \to S_4$$
$$C: S_5 \times S_3 \to S$$
$$D: S_1 \times S_2 \to S_5$$

THEOREM 4.1. *The general solution of* (1)*, where A, B, C, D are G-groupoids, is given by* (2)*, where:*

(i) $P: S_2 \times S_3 \to S_4$ *and* $Q: S_1 \times S_2 \to S_5$ *are arbitrary functions (G-groupoids)*

(ii) $f: S_4 \to S^{S_1}$ *and* $g: S_5 \to S^{S_3}$ *are functions such that* (3) *holds, where* $\lambda_{xy}(\rho_{xy})$ *is left (right) translation of an arbitrary function* $T: S_1 \times S_2 \times S_3 \to S$*, satisfying* (4)*. (Obvious adaptations shoulds be made in definitions of* $\square$, λ_{xy}, ρ_{xy}*).*

Again, the proof of Th 4.1 is similar to the proof of Th 5.

In the same way as before, we can prove the following theorem of Milić [5]:

THEOREM 4.2. *If GD-groupoids A, B, C, D satisfy* (1) *and* A_2, C_1 *are bijections for some* $p, r \in S$*, then the general solution of* (1) *is given by* (7)*, where* $\cdot$ *is an arbitrary group on* S, A_1, B_1, B_2, C_2, D_1, D_2 *arbitrary functions and* A_2, C_1 *arbitrary bijections such that:*

$$A_1 = C_1 D_1 \quad A_2 B_1 = C_1 D_2 \quad A_2 B_2 = C_2.$$

EXAMPLE 5. If $|S| = 2$, then there are exactly 1344 solutions of (1) on S. The proof is given in [6], along with all solutions.

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