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# Δ-ENDOMORPHISM NEAR-RINGS

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The concept of a distributively generated near-rings arise if we define addition and multiplication of endomorphisms of the group $(G,+)$ in the usal manner. It is possibble to consider the set of the mappings of $(G,+)$ into itself which are similar to the endomorphisms of a group in such a way that their "linearity" is corrected by the elements from a normal subgroup $\Delta$ of the group $(G,+)$. These mappings are called $\Delta$-endomorphisms of $(G,+)$. The set of $\Delta$-endomorphisms of $G$ generate (additively) a near-ring $\mathcal{E}_\Delta(G)$, whose defect depends on the shoice of the subgroup $\Delta$. Also, $\Delta$-endomorphisms for which is invariant every fully invariant subgroup of the group $(G,+)$, are investigated. In this case we obtain the subnearring $E_\Delta(G)$ of the near-ring $\mathcal{E}_\Delta(G)$. Some known properties of the endomorphism near-rings were transfered to the $\Delta$-endomorphism near-rings.

Some elementary results relating to the $E_\Delta$-invariant subgroups of $(G,+)$ are presented in Section 2. In Secton 3 we consider the structure of ideals of the near-ring $E_\Delta(G)$, generalizing the results which were obtained by $H$. Johnson in [8] and [9] for the near-ring of endomorphisms. The result in Section 4 refers to the problem embedding of near-rings into some near-ring of $\Delta$-endomorphisms and generalizes the Theorem Heatherly and Malone in [7]. Also, a $\mathcal{D}$-direct sum of subnear-rings of the near-ring $E_\Delta(G)$ is considered, where $\mathcal{D}$ is a defect of $E_\Delta(G)$.

## 1. Preliminaries

Throughout this paper term "near-ring" shall mean "left near-ring" $R$ satisfying $ox = o$ for all $x \in R$. The necessary definitions concerning near-rings with a defect of distributivity are now given.

A set of generators of the near-ring $R$ is a multiplicative subsemigroup $S$ of $R$ whose elements generale $(R,+)$. Let $S$ be a set of generators of the near-ring $R$ and let

$$D_S = \{d\colon d = -(xs+ys)+(x+y)s, \quad x,y \in R, \quad s \in S\}.$$

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The normal subgroup $D$ of the group $(R,+)$ which is generated by the set $D_S$ is called the defect of distributivity of the near-ring $R$. Thus, for all $x, y \in R$ and $s \in S$ there exists $d \in D$ such that

$$(x+y)s = xs + ys + d.$$

The near-ring $R$ with the defect $D$ will be detoned by $(R,S)$ when we wish to stress the set of generators $S$. A near-ring $R$ is called $D$-distributive if $R = S$, i.e. for each $x, y, z \in R$ there exists $d \in D$ such that

$$(x+y)z = xz + yz + d.$$

Let $(R,S)$ be a near-ring with the defect $D$ and $A \subset R$. The normal subgroup $\bar{A}$ of $(R,+)$ generated by the set $A \cup AS$ has the elements of the form

$$\bar{a} = \sum_i (r_i \pm a_i s_i + m_i {a_i}' - r_i), \quad (r_i \in R,\ \ a_i, {a_i}' \in A,\ \ s_i \in S,\ \ m_i\text{– integers}).$$

For all $r$, $r_i \in R$, $a_i$, ${a_i}' \in A$ and $s$, $s_i \in S$ there exists $d_1$, $d_2 \in D$ such that

$$(r+\bar{a})s = rs + \bar{a}s + d_1 = rs + \left(\sum_i (r_i \pm a_i s_i + m_i {a_i}' - r_i)\right) s + d_1$$

$$(r+\bar{a})s = \sum_i (r_i s \pm a_i s_i s + m_i {a_i}' s - r_i s) + d_2 + d_1.$$

The normal subgroup $D_r$ of the group $(R,+)$ generated by the elements $d_2 + d_1 = d \in D$ which have been obtained in the previous manner, is called a relative defect of the subset $A$ with respect to $R$. It is obvious that $D_r \subseteq D$.

Lemma 1.1. ([4]. *Lemma* 3.2*) Let $(R,S)$ be a near-ring with defect. The normal subgroup $B$ of the group $(R,+)$ is a right ideal of $R$ if and only if $B$ is an $S$-subgroup which contains the relative defect of the subset $B$ with respect to $R$.*

Proposition 1.2. ([5], *Coroll. of Lemma* 1.1*) Let $(R,S)$ be a near-ring with defect and $A \subset R$. The normal subgroup $\bar{A}$ of $(R,+)$ generated by $A \cup RA \cup AS \cup RAS$ is an ideal of $R$ if and only if $\bar{A}$ contains the relative defect of the subset $A \cup RA$ with respect to $R$.*

Proposition 1.3. ([4], *Theorem* 2.3 *b) Every direct sum of the near-rings $R_i$ with the defect $D_i$ respectively, is a near-ring $R$ whose defect is a direct sum of the defects $D_i$.*

## 2. Elementary properties of Δ-endomorphisms

Let $M_0(G)$ be a set of zero preserving mappings of the group $(G,+)$ into itself.

Definition. Let $\Delta$ be a normal subgroup of the group $(G,+)$. The mapping $f \in M_0(G)$ with $(\Delta)f \subseteq \Delta$ is called $\Delta$-endomorphism of the group $(G,+)$ if for all $x, y \in G$ there exists $\delta \in \Delta$ such that

$$(x+y)f = (x)f + (y)f + \delta.$$

It is easy to prove by induction that for each $x_1, \dots, x_n \in G$ and some $\Delta$-endomorphism $f$ there exists $\delta \in \Delta$ such that

$$(x_1 + \ldots + x_n)f = (x_1)f + \cdots + (x_n)f + \delta.$$

In the case $\Delta = (0)$ we obtain the endomorphisms of the group $(G,+)$. The set of all $\Delta$-endomorphisms of the group $(G,+)$ will be denoted by $\mathcal{E}nd_\Delta(G)$. This set is a semigroup with respect to composition.

Let us denote by $(G,\Delta)_0$ the set of all mappings $h$: $G \to \Delta$ with $(0)h = 0$. It is clear that $(G,\Delta)_0 \subseteq \mathcal{E}nd_\Delta(G)$. Thus, for $\Delta \neq (0)$ it follows that $\mathcal{E}nd_\Delta(G) \neq End(G)$.

If $(G,+)$ is non-commutative, then the set of all $\Delta$-endomorphisms of $G$ will not be closed under pointwise addition. However, the set of all (finite) sums and differences of $\Delta$-endomorphisms of $G$ forms a near-ring, which will be designated by $\mathcal{E}_\Delta(G)$. Namely, if $f = \sum_i(\pm t_i)$ and $h = \sum_j(\pm t_j{}')$, $(t_i, t_j{}' \in \mathcal{E}nd_\Delta(G))$, then for all $x \in G$ we have

$$\begin{aligned}(x)fh &= \sum_j \pm \left(\sum_i ((\pm x)t_i)\right) t_j{}' \\ &= \sum_j \pm \left(\sum_i (\pm x)t_i t_j{}' + \delta_{ij}\right) \\ &= \sum_j \pm \left(\sum_i (\pm x)t_i t_j{}'\right) + \delta, \quad (\delta_{ij}, \delta \in \Delta).\end{aligned}$$

But, the element $\delta \in \Delta$ depends on $x$. If we put $\delta = (x)\alpha$, then $\alpha \in (G,\Delta)_0$ i.e. $\alpha \in \mathcal{E}nd_\Delta(G)$. Hence,

$$(x)fh = (x)\left[\left(\sum_j \left(\pm \sum_i t_i t_j{}'\right)\right) + \alpha\right], \text{ i.e.}$$

$$fh = \sum_j \left(\sum_i (\pm t_{ij})\right) + \alpha,$$

where $t_i t_j{}' = t_{ij} \in \mathcal{E}nd_\Delta(G)$ and $\alpha \in \mathcal{E}nd_\Delta(G)$.

The normal subgroup $\mathcal{D}$ of the group $(\mathcal{E}_\Delta(G),\ +)$ generated by

$$\{\delta : \delta = -(ht+ft)+(h+f)t, \quad h, f \in \mathcal{E}_\Delta(G),\ \ t \in \mathcal{E}nd_\Delta(G)\}$$

is a defect of distributivity of the near-ring $\mathcal{E}_\Delta(G)$. It is clear that $\mathcal{D} \subseteq (G,\Delta)_0$. For example, the near-ring $\mathcal{E}_\Delta(Z_4) = \{f_0, f_1, \dots, f_{15}\}$, where $\Delta = \{0,2\}$, has the defect $\mathcal{D} = \{f_0, f_3, f_{12}, f_{13}\}$ (table 1).

If the commutator subgroup $G'$ of $(G,+)$ is a subset of $\Delta$, then $\mathcal{E}_\Delta(G)$ is a $\mathcal{D}$-distributive near-ring, where $\mathcal{D}$ is the defect of $\mathcal{E}_\Delta(G)$. Let $G$ be a nilpotent group and $\Delta$ its maximal subgroup. Then by Corollary 10.3.2 of [6] it follows that the near-ring $\mathcal{E}_\Delta(G)$ is $\mathcal{D}$-distributive, where $\mathcal{D}$ is the defect of $\mathcal{E}_\Delta(G)$.

Let $(R,S)$ be a near-ring with the defect $D$. For all $s \in S$ and $x \in R$ there is a map $f_s : x \to xs$ from $R$ into $R$. These maps are $D$-endomorphisms. Let us denote by $\mathcal{E}_D(R)$ the near-ring of "right multiplications" of the near-ring $R$ with the defect $D$. The defect of distributivity of $\mathcal{E}_D(R)$ is the set

$$\{f_d : (x)f_d = xd,\ \ x \in R,\ \ d \in D\}.$$

Proposition 2.1. *If $\Delta$ is a proper normal subgroup of the group $(G,+)$, then $\mathcal{E}_\Delta(G) \subset M_0(G)$.*

Proof. Anyhow $\mathcal{E}_\Delta(G) \subseteq M_0(G)$. If $(0) \neq \Delta \neq G$ and $y \in G \backslash \Delta$, then the map $h \in M_0(G)$ can be defined as follows

$$x(h) = \begin{cases} y, & x \in \Delta,\, x \neq 0 \\ 0 & x = 0 \\ x, & x \notin \Delta \end{cases}$$

Since $(\Delta)\ \mathcal{E}_\Delta(G) \subseteq \Delta$, we have $h \notin \mathcal{E}\Delta(G)$.

If $B$ is a fully invariant subgroup of the group $(G,+)$, then $B$ must not be invariant with respect to all $\Delta$-endomorphisms of $(G,+)$. For example, the subgroup $B = \{0,2,4\}$ of $(Z_6,+)$ is not invariant with respect to the $\Delta$-endomorphism $f = \binom{012345}{003003}$, where $\Delta = \{0,3\}$.

Let $\Delta$ be a proper normal subgroup of the group $(G,+)$. There exist nontrivial $\Delta$-endomorphisms for which are invariant all subgroups of $(G,+)$. For instance, the mapping $f \in M_0(G)$ with $(x)f = x$ for all $x = \Delta$, and $(x)f = 0$ for all $x \in G \backslash \Delta$ is such a $\Delta$-endomorphism. Let us denote by $End_\Delta(G)$ the biggest subsemigroup of the semigroup $\mathcal{E}nd_\Delta(G)$ for which are invariant all fully invariant subgroups of the group $(G,+)$. If we denote by $E_\Delta(G)$ the additive group generated by $\mathcal{E}nd_\Delta(G)$, then $E_\Delta(G)$ is a near-ring whose set of generators $End_\Delta(G)$ is contained in a set of generators $\mathcal{E}nd_\Delta(G)$ of the near-ring $\mathcal{E}_\Delta(G)$. Every fully invariant subgroup of $(G,+)$ which is invariant with respect to $End_\Delta(G)$, is invariant with respect to

$E_\Delta(G)$ as well. For this reason we say that the subgroups of this kind are $E_\Delta$-invariant.

EXAMPLE 1. The group $(Z_6,+)$ has 96 $\Delta$-endomorphisms for which only the subgroup $\Delta = \{0,3\}$ is invariant. However, the set $End_\Delta(Z_6) = \{f_0, f_1, \dots, f_{23}\}$ contains all $\Delta$-endomorphisms of $(Z_6,+)$ for which both subgroups $\Delta$ and $B = \{0,2,4\}$ are invariant (table 2). If we take for $\Delta$ the subgroup $B$, then there exist 486 $\Delta$-endomorphisms. But by claiming that both subgroups of $(Z_6,+)$ are invariant this number will be reduced to 54.

If $\Delta$ is a fully invariant subgroup of $(G,+)$, then a near-ring $E_\Delta(G)$ contains the endomorphism near-ring $E(G)$. A several following propositions are related to the elementary proposities of $E_\Delta$-invariant subgroup and they generalize the corresponding results of M. Jonson in [8].

PROPOSITION 2.2. *Let $\Delta$ be a fully invariant subgroup of $(G,+)$ and let $y \in G$, $(y \neq 0)$. If $\mathcal{H}$ is a right $E_\Delta(G)$-subgroup, then $(y)\mathcal{H}$ is $E_\Delta$-invariant subgroup of $(G,+)$.*

The proof is quite analagous with that in ([8], Lemma 3.1).

COROLLARY. Let $B$ be $E_\Delta$-invariant subgroup of $(G,+)$ and let $y \in B$, $(y \neq 0)$. If $\mathcal{H}$ is a right $E_\Delta(G)$-subgroup, then $(y)\mathcal{H}$ is $E_\Delta$-invariant subgroup of $(G,+)$.

DEFINITION. Let $B$ be a subgroup of the group $(G,+)$ and $\mathcal{H} \subseteq M_0(G)$. If $B$ is an invariant subgroup with respect to $\mathcal{H}$, then we say that $\mathcal{H}$ acts transitively on $B$ if for all $x \in \mathbf{B}$, $(x \neq 0)$ we have $(x)\mathcal{H} = B$.

DEFINITION. The group $(G,+)$ is called $E_\Delta$-simple if and only if $(G,+)$ has not proper $E_\Delta$-invariant subgroups.

Using Corollary of Proposition 2.2 we obtain the following.

PROPOSITION 2.3. *Let $B$ be an $E_\Delta$-invariant subgroup of the group $(G,+)$. Then $B$ is a minimal $E_\Delta$-invariant subgroup of $(G,+)$ if and only if $E_\Delta(G)$ acts transitively on $B$.*

COROLLARY. Let $\Delta$ be a fully invariant subgroup of $(G,+)$. $E_\Delta(G)$ acts transitively on $G$ if and only if $G$ is $E_\Delta$-simple.

Let $G$ be a group and $B \subset G$. Denote by $\mathcal{A}(B)$ a right annihilator of $B$ in $E_\Delta(G)$, that is, $\mathcal{A}(B) = \{f \in E_\Delta(G)\colon\ (b)f = 0 \text{ for all } b \in B\}$.

PROPOSITION 2.4. *Let $B_i$ $(i \in I)$ be a collection of minimal $E_\Delta$-invariant subgroups of the group $(G,+)$ and let $\mathcal{N}$ be a right $E_\Delta(G)$-subgroup of $E_\Delta(G)$ containing only nilpotent elements. Then $\mathcal{N} \subseteq \cap_i \mathcal{A}(B_i)$.*

PROOF. Let $h \in \mathcal{N}$ and suppose that for some $b \in B_p$ $(p \in I)$, $(b)h \neq 0$. By Proposition 2.2 $(b)hE_\Delta(G)$ is $E_\Delta$-invariant subgroup. Since $B_p$ is a minimal $E_\Delta$-invariant subgroup of $(G,+)$, there exists $f \in E_\Delta(G)$ such that $(b)hf = b$. Hence

$hf$ is not nilpotent. On the other hand, $hf \in \mathcal{N}$ and this contradiction establishes the proposition.

The next proposition is easily verified.

PROPOSITION 2.5. *Let $B_i$ $(i \in I)$ be a collection of $E_\Delta$-invariant subgroups of the group $(G,+)$. If $\Delta \subseteq \sum_i B_i$ then $\sum_i B_i$ is $E_\Delta$-invariant subgroup.*

### 3. The ideal structurs of $E_\Delta(G)$

The results in this section refer to the ideal structures of the near-ring $E_\Delta(G)$. The results of M. Johnson ([8], Lemmas 6.1, 8.5, Thms 6.2, 6.11, 6.12, Propositions 8.9, 8.15) and ([9], Lemma 11, Thms 8 and 16) become a special case of these, when we take an endomorphism near-ring $E(G)$ instead $E_\Delta(G)$.

If $\mathcal{H}$ is a subset of $E_\Delta(G)$, we define

$$\Im(\mathcal{H}) = \{(x)h : x \in G, \quad h \in \mathcal{H}\}.$$

Obvious, $\Im(\mathcal{D}) \subseteq \Delta$, where $\mathcal{D}$ is the defect of the near-ring $E_\Delta(G)$.

PROPOSITION 3.1. *Let $B$ be an $E_\Delta$-invariant subgroup of the group $(G,+)$. If $\Im(\mathcal{D}_r) \subseteq B$, where $\mathcal{D}_r$ is the relative defect of the subset $\mathcal{B} = \{f \in E_\Delta(G) : \Im(f) \subseteq B\}$ with respect to $E_\Delta(G)$, then $\mathcal{B}$ is an ideal of $E_\Delta(G)$.*

PROOF. It is easy to show that $\mathcal{B}$ is a normal subgroup of $(E_\Delta(G),+)$ and $E_\Delta(G)$-subgroup of $E_\Delta(G)$. If $\delta \in \mathcal{D}_r$ then $\delta \in \mathcal{B}$ because $\Im(\mathcal{D}_r) \subseteq B$. Hence $\mathcal{B}$ contains the relative defect of the subset $\mathcal{B}$ with respect to $E_\Delta(G)$. Therefore, by Lemma 1.1 it follows that $\mathcal{B}$ is a right ideal of $E_\Delta(G)$. Also, $\mathcal{B}$ is a lift $E_\Delta(G)$-subgroup. Thus $\mathcal{B}$ is an ideal of $E_\Delta(G)$.

PROPOSITION 3.2. *Let $\Delta \neq G$ be a nonzero fully invariant subgroup of the group $(G,+)$. Then $E_\Delta(G)$ is not a simple near-ring.*

PROOF. Let $\mathcal{D}_r$ be a relative defect of the subset

$$\mathcal{B} = \{f \in E_\Delta(G) : \Im(f) \subseteq \Delta\}$$

with respect to $E_\Delta(G)$. Because $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G,\Delta)_0$, we have $\Im(\mathcal{D}_r) \subseteq \Delta$. By Proposition 3.1, $\mathcal{D}$ is an ideal of $E_\Delta(G)$. Since $\Delta \neq G$ it follows that the identity map is not in $\mathcal{B}$, i.e. $\mathcal{B} \neq E_\Delta(G)$. Let us define the map $h \in (G,\Delta)_0$ as follows

$$(x)h = \begin{cases} x, & x \in \Delta \\ 0, & x \notin \Delta \end{cases}$$

This map is a nonzero $\Delta$-endomorphism and $\Im(h) \subseteq \Delta$, i.e. $h \in \mathcal{B}$. Hence, $\mathcal{B}$ is a proper ideal of $E_\Delta(G)$.

PROPOSITION 3.3. *Let $\Delta$ be a fully invariant subgroup of the group $(G,+)\cdot$ $E_\Delta(G)$ is simple if and only if $G$ is $E_\Delta$-simple.*

PROOF. If $G$ is a nonzero $E_\Delta$-simple group it must be either $\Delta = (0)$ or $\Delta = G$. For $\Delta = (0)$ the results follows from ([8], Th. 6.12) and for $\Delta = G$ it follows from ([2], Lemma 4).

Conversely, let now $E_\Delta(G)$ be a simple near-ring. If $\Delta = (0)$ the result follows from ([8], Th. 6.12). If $\Delta \neq (0)$ then it is not a proper subgroup of $G$. Namely, if $\Delta \neq G$ then by Proposition 3.2 $E_\Delta(G)$ is not a simple near-ring. Thus, let $\Delta = G$, i.e. $E_\Delta(G) = H_0(G)$. If $B$ is a proper subgroup of $(G, +)$, then there always exists $f \in M_0(G)$ for that $B$ is not invariant. Therefore, $G$ is an $E_\Delta$-simple group.

THEOREM 3.4. *If $B$ is a sum of all minimal nozero $E_\Delta$-invariant subgroups of a finite group $(G, +)$ and $\Delta \subseteq B$ is fully invariant subgroup of $(G, +)$, then $\mathcal{B} = \{h \in E_\Delta(G) \colon \Im(h) \subseteq B\}$ is a proper nonzero ideal of $E_\Delta(G)$.*

PROOF. By Proposition 2.5 it follows that $B$ is $E_\Delta$-invariant subgroup. If $\mathcal{D}_r$ is a relative defect of the subset $\mathcal{B}$ with respect to $E_\Delta(G)$, then $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G, \Delta)_0$. Since, $\Delta \subseteq B$ we have $\Im(\mathcal{D}_r) \subseteq B$. Thus, by Proposition 3.1 $\mathcal{B}$ is an ideal of $E_\Delta(G)$. Clearly, $\mathcal{B} \neq E_\Delta(G)$. Let $\{x_1, \dots, x_n\} = G$. By Proposition 2.2 $(x_p)E_\Delta(G)(p = 1, \dots, n)$ is $E_\Delta$-invariant subgroup of $(G, +)$. Thus, $(x_p)E_\Delta(G) \cap B \neq (0)$ for all $p = 1, \dots, n$. Now the proof is similar to the proof of the Theorem 6.2 in [8].

PROPOSITION 3.5. *Let $B$ be a sum of all minimal nonzero $E_\Delta$-invariant subgroups of a finite group $(G, +)$ and let $\Delta \subseteq B$ be a fully invariant subgroup of $(G, +)$. If $\mathcal{H}$ is a minimal right $E_\Delta(G)$-subgroup of $E_\Delta(G)$ then $\Im(\mathcal{H}) \subseteq B$.*

*The proof is the same as that in ([9], Proposition 6.)*

THEOREM 3.6. *Let $B$ a minimal nonzero $E_\Delta$-invariant subgroup of the group $(G, +)$. If $b \in B (b \neq 0)$, then $\mathcal{A}(b)$ is a maximal right ideal of $E_\Delta(G)$.*

PROOF. If $\Delta = G$ then $E_\Delta(G) = M_0(G)$. In this case the result follows from ([10], Th. 3). If $\Delta = (0)$ then result follows by Lemma 8.5 of [8]. Let now $\Delta \neq (0)$ and $\Delta \neq G$. Since $e \notin \mathcal{A}(b)$ ($e$ is the identity map), we have that $\mathcal{A}(b) \neq E_\Delta(G)$. Let us suppose that there is a right ideal $\mathcal{P}$ of $E_\Delta(G)$ such that $\mathcal{A}(b)$ is a proper subset of $\mathcal{P}$. By Corollary of Proposition 2.2 it follows that $(b)\mathcal{P}$ is an $E_\Delta$-invariant subgroup of $(G, +)$. Thus, either $(b)\mathcal{P} = B$ or $(b)\mathcal{P} = (0)$, because $B$ is a minimal $E_\Delta$-invariant subgroup. Since $\mathcal{A}(b) \subset \mathcal{P}$ we have $(b)\mathcal{P} = B$. Consequently, there exists $f \in \mathcal{P}$ such that $(b)f = b$. Let $h = -f + e$, where $e$ is the identity map of $G$ itself. Clarly $h \in \mathcal{A}(b)$. Thus, $e = h + f \in \mathcal{P}$ and $\mathcal{P} = E_\Delta(G)$. Therefore, $\mathcal{A}(b)$ is a maximal ideal of $E_\Delta(G)$.

THEOREM 3.7. *Let $B$ be a minimal nonzero $E_\Delta$-invariant subgroup of the group $(G, +)$. Then $\mathcal{A}(B)$ is a maximal ideal of $E_\Delta(G)$.*

The proof is similar to the proof of the Proposition 8.15 in [8].

EXAMPLE 2. Let $E_\Delta(Z_6)$ be a near-ring of $\Delta$-endomorphisms of the group $(Z_6, +)$ (table 2). The subgroups $B_1 = \Delta = \{0, 3\}$ and $B_2 = \{0, 2, 4\}$ of $(Z_6, +)$

are minimal $E_\Delta$-invariant subgroups. The annihilator ideals

$$\mathcal{A}(B_1) = \{f_0, f_2, f_4, f_6, f_7, f_9, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$$

and

$$\mathcal{A}(B_2) = \{f_0, f_3, f_9, f_{11}, f_{12}, f_{13}, f_{14}, f_{21}\}$$

are maximal ideals of $E_\Delta(Z_6)$.

The following theorem gives another type of a maximal right ideal of $E_\Delta(G)$ and generalizes the Proposition 8.9 in [8].

THEOREM 3.8. *Let $B$ be a maximal $E_\Delta$-invariant subgroup of a finite group $(G,+)$ and let $\Delta \subseteq B$ be a fully invariant subgroup of $(G,+)$. If $x \in G\backslash B$ then $\mathcal{B} = \{\beta \in E_\Delta(G) : (x)\beta \in B\}$ is a maximal right ideal of $E_\Delta(G)$.*

PROOF. It is easy to show that $\mathcal{B}$ is a normal $E_\Delta(G)$-subgroup. Let $\mathcal{D}_r$ be a relative defect of the subset $\mathcal{B}$ with respect to $E_\Delta(G)$. Since $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G,\Delta)_0$ we have $\mathcal{D}_r \subseteq \mathcal{B}$. Thus, by Lemma 1.1 it follows that $\mathcal{B}$ is a right ideal of $E_\Delta(G)$. Morover, $\mathcal{B} \neq E_\Delta(G)$, because $\mathcal{B}$ contains no the identity map $e$ of $G$ into itself.

We will prove that $\mathcal{B}$ is a maximal right ideal of $E_\Delta(G)$. Let $\mathcal{P}$ be a right ideal of $E_\Delta(G)$ such that $\mathcal{B} \subset \mathcal{P}$. Assume that $\alpha \in \mathcal{P}$ and $\alpha \notin \mathcal{B}$ i.e. $(x)\alpha \notin B$. The normal subgroup $(x)\alpha E_\Delta(G) + B$ is $E_\Delta$-invariant. Namely, for all $f \in E_\Delta(G)$ and $t \in End_\Delta(G)$ we have

$$((x)\alpha f + b)\, t = (x)\alpha f t + (b)t + \delta \in (x)\alpha E_\Delta(G) + B,$$

because $\delta \in \Delta \subseteq B$ and $b,\ (b)t \in B$. Since $B$ is a maximal $E_\Delta$-invariant subgroup of $(G,+)$, then $(x)\alpha E_\Delta(G) + B = G$. Thus, there exist $f \in E_\Delta(G)$ and $b \in B$ such that $(x)\alpha f + b = x$. The map $h: G \to G$ with $h = -\alpha f + e$ belongs to $E_\Delta(G)$. Since $(x)h = -(x)\alpha f + x = b - x + x = b \in B$ we hawe $h \in \mathcal{B}$, i,e. $h \in \mathcal{P}$. Also, $\alpha f \in \mathcal{P}$. Hence $e = (\alpha f + h) \in \mathcal{P}$ and $\mathcal{P} = E_\Delta(G)$. Therefore, $\mathcal{B}$ is a miximal right ideal of $E_\Delta(G)$.

EXAMPLE 3. Let $E_\Delta(Z_4)$ be a near-ring of $\Delta$-endomorphisms of the group $(Z_4,+)$ (table 1). The subgroup $\Delta = \{0,2\}$ is a maximal $E_\Delta$-invariant subgroup of $(Z_4,+)$. For $x = 3 \notin \Delta$ the set

$$\mathcal{B} = \{f \in E_\Delta(Z_4) : (3)f \in \Delta\} = \{f_0, f_3, f_7, f_8, f_{12}, f_{13}, f_{14}, f_{15}\}$$

is a maximal right ideal of $E_\Delta(Z_4)$.

THEOREM 3.9 *Let $B \neq G$ be a sum of all minimal monzero $E_\Delta$-invariant subgroups of a finite group $(G,+)$. If $\Delta \subseteq B$ is a fully invariant subgroup of $(G,+)$ then the nil radical of $E_\Delta(G)$ is nonzero.*

PROOF. Let $B_i$ $(i \in I)$ be a collection of all minimal nonzero $E_\Delta$-invariant subgroups of $(G,+)$ and let $\mathcal{A}(B_i)$ be annihilator ideals of the subgroups $B_i$ $(i \in I)$.

We prove first that $\cap_i \mathcal{A}(B_i)$ is nonzero. Suppose, if possible $\cap_i \mathcal{A}(B_i) = (0)$. By using the Proposition 2.4 it follows that $E_\Delta(G)$ contains no nonzero right $E_\Delta(G)$-subgroup consisting of nilpotent elements. Thus, by Theorem 3 of [3] $E_\Delta(G)$ is a direct sum of minimal nonzero $E_\Delta(G)$-subgroups. Hence, by Proposition 3.5 we obtain $\Im(E_\Delta(G)) \subseteq B$. In particular, for identitety map $e \in E_\Delta(G)$ we have $G = (G)e \subseteq B$, i.e. $G = B$. But this contradictory to the supposition that $G \neq B$. Therefore $\cap_i \mathcal{A}(B_i) \neq (0)$. Since the nil radical is the sum of all nil ideals and $\cap_i \mathcal{A}(B_i)$ is nonzero nil ideal, it follows that the nil radical of $E_\Delta(G)$ is nonzero.

PROPOSITION 3.10. *Let $\Delta$ be a minimal fully invariant subgroup of a finite group $(G,+)$ and let $\mathcal{N}$ be any nilpotent $E_\Delta(G)$-subgroup of $E_\Delta(G)$. If the normal subgroup $\mathcal{W}$ of the group $(E_\Delta(G),+)$, generated by the set $E_\Delta(G)\mathcal{N}$, contains the relative defect of the subset $E_\Delta(G)\mathcal{N}$ with respect to $E_\Delta(G)$, then $\mathcal{W}$ is a nilpotent ideal of $E_\Delta(G)$.*

PROOF. By Proposition 1.2 $\mathcal{W}$ is an ideal of $E_\Delta(G)$. Since $\mathcal{N}$ is a right $E_\Delta(G)$-subgroup of $E_\Delta(G)$ and $E_\Delta(G)$ has identity, the elements of $\mathcal{W}$ have the form $w = \sum_i (f_i \pm h_i n_i - f_i)$, $(f_i,\ h_i \in E_\Delta(G),\ n_i \in \mathcal{N})$. If $x \in G$, $x \neq 0$, and $n \in \mathcal{N}$, then $E_\Delta$-invariant subgroup of $(G,+)$ generated by $(x)n$ is properly contained in the $E_\Delta$-invariant subgroup generated by $x$. Indeed, let $X$ be $E_\Delta$-invariant subgroup generated by $x$ and let $Y$ be $E_\Delta$-invariant subgroup generated by $(x)n$. Clearly $Y \subseteq X$. Let us suppose that $Y = X$. Then there exists $f \in E_\Delta(G)$ such that $(x)nf = x$ and, we have a contradiction, because $nf \in \mathcal{N}$ and $\mathcal{N}$ is a nilpotent $E_\Delta(G)$-subgroup. Thus $Y \subset X$.

Let $B = \sum_k B_k$ be a sum of all minimal $E_\Delta$-invariant subgroups of $(G,+)$ and let $w = \sum_i (f_i \pm h_i n_i - f_i) \in \mathcal{W}$, $(f_i, h_i \in E_\Delta(G),\ n_i \in \mathcal{N})$. Then there exists a positive integer $p$ such that $(x)w^p \in B$, because every fully invariant subgroup generated by $(x)h_i n_i$ is properly contained in the fully invariant subgroup generated by $(x)h_i$. Thus,

$$(x)w^{p+1} = ((x)w^p)w = \left(\sum_k b_k\right) w$$
$$= \sum_i \left[ \left(\sum_k b_k\right) f_i \pm \left(\sum_k b_k\right) h_i n_i - \left(\sum_k b_k\right) f_i \right].$$

By Proposition 2.5 $B$ is $E_\Delta$-invariant subgroup, i.e.

$$\left(\sum_k b_k\right) h_i n_i = \left(\sum_k {b_k}'\right) n_i, \quad (b_k, {b_k}' \in B_k).$$

Let $n_i = \sum_j (\pm t_{ij})$, $(t_{ij} \in End_\Delta(G))$, then

$$\left(\sum_k b_k\right) h_i n_i = \left(\sum_k {b_k}'\right) n_i = \left(\sum_k {b_k}'\right) \sum_j (\pm t_{ij}) =$$
$$= \sum_j \pm \left(\sum_k {b_k}'\right) t_{ij} = \sum_j \pm \left(\sum_k ({b_k}') t_{ij}\right) + \delta, \ (\delta \in \Delta).$$

The elements of different minimal $E_\Delta$-invariant subgroups $B_k$ commute elementwise. Thus

$$\left(\sum_k b_k\right) h_i n_i = \sum_k \left[ ({b_k}') \sum_j (\pm t_{ij}) \right] + \delta = \sum_k ({b_k}') n_i + \delta.$$

Therefore

$$(x)w^{p+1} = \sum_i \left[ \left(\sum_k b_k\right) f_i \pm \left(\sum_k ({b_k}')n_i + \delta\right) - \left(\sum_k b_k\right) f_i \right].$$

By Proposition 2.4, $n_i \in \mathcal{A}(B_k)$ for all $k$ and hence $(x)w^{p+1} \in \Delta$. Thus, there exist $\delta', \delta'' \in \Delta$ such that

$$\begin{aligned} (x)w^{p+2} &= ((x)w^{p+1})w = (\delta')w = (\delta') \sum_i (f_i \pm h_i n_i - f_i) = \\ &= \sum_i [(\delta')f_i \pm (\delta')h_i n_i - (\delta')f_i] = \\ &= \sum_i [(\delta')f_i \pm (\delta'')n_i - (\delta')f_i] = 0. \end{aligned}$$

Thus, every element $w \in \mathcal{W}$ is nilpotent. Because $G$ is finite it follows that $\mathcal{W}$ is nilpotent.

THEOREM 3.11. *Let $\Delta$ be a minimal fully invariant subgroup of a finite group $(G,+)$ and let $\mathcal{N}$ be any nilpotent $E_\Delta(G)$-subgroup of $E_\Delta(G)$. If the normal subgroup $\omega$ of the group $(E_\Delta(G),+)$ generated by the set $E_\Delta(G)\mathcal{N}$ contains the relative defect of the subset $E_\Delta(G)\mathcal{N}$ with respect to $E_\Delta(G)$, then the nil radical $\eta(E_\Delta(G))$ coincides with the radical $J_2(E_\Delta(G))$.*

PROOF. By Proposition 3.10 $\mathcal{N} \subseteq \eta(E_\Delta(G))$, because the nil radical $\eta(E_\Delta(G))$ is the sum of all nil ideals. Thus, $E_\Delta(G)/\eta(E_\Delta(G))$ contains no nonzero nilpotent right $E_\Delta(G)$-subgroups. By using two theorems of Blackett ([3], Thms 1 and 2) it follows that every minimal right ideal of $E_\Delta(G)/\eta(E_\Delta(G))$ contains an idempotent element. By Beidleman [1], a proper ideal $B$ of a near-ring $R$ is called a strong radical-ideal of $R$ if and only if every nonzero right ideal $R/B$ contains a minimal right ideal which contains an idempotent element. Hence, $\eta(E_\Delta(G))$ is a strong radical-ideal of $E_\Delta(G)$. The following step in the proof is the same as that of ([1], Th. 8).

If the group $(G,+)$ is equal to the sum of its minimal fully invariant subgroups, then as an immediate consequence of Proposition 3 of [1], $J_2(E(G)) = (0)$, where $E(G)$ is an endomorphism near-ring. However, this is not true for near-ring $E_\Delta(G)$ if $(G,+)$ is equal to the sum its minimal $E_\Delta$-invariant subgroups, where $\Delta$ is a proper minimal $E_\Delta$-invariant subgroup of $(G,+)$. For example, the group $(Z_6,+)$

is a direct sum of a minimal $E_\Delta$-invariant subgroups $B_1 = \Delta = \{0,3\}$ and $B_2 = \{0,2,4\}$, but the radical

$$J_2(E_\Delta(Z_6)) = \mathcal{D} = \{f_0, f_9, F_{12}, f_{14}\} \neq (0),$$

where $\mathcal{D}$ is the defect of the near-ring $E_\Delta(Z_6)$ (table 2). In general, let $(G,+)$ be a direct sum of minimal $E_\Delta$-invariant subgroups, where $\Delta$ is a proper $E_\Delta$-invariant subgroup and let $\mathcal{D}$ be the defect of the near-ring $E_\Delta(G)$. Is it $J_2(E_\Delta(G)) = \mathcal{D}$? The answer is connected to the posiibility that every $\Delta$-endomorphism $f$ of $(G,+)$ can be uniqeuly expressed in the form $f = h + \delta$, where $h \in E(G)$ and $\delta \in \mathcal{D}$.

### 4. Embeddings of near-ring with defect into some $E_\Delta(G)$

The problem of embedding the near-rings with the defect of distributivity is not easy. The following results refer to the particular case and generalize corresponding results for distributively generated near-ring (see [7]).

By using the technique of "right multiplicator" we have.

Proposition 4.1. *Let $(R,S)$ be a near-ring with the defect $D$. If $A(R) = (0)$, then $R$ embeos in $E_D(R)$.*

Proposition 4.2. *Let $R$ be a near-ring such that $R = A(R) \oplus B$, where $B$ is an ideal of $R$. Let $D \neq R$ be the defect of distributivity of $R$. Then $D$ is the defect of the near-ring $B$.*

Proof. Since $B \simeq R/A(R)$ it follows that $B$ is a near-ring with the defect $D'$. On the other hand $A(R) = \{a \in R\colon ra = 0, \text{ for all } r \in R\}$, i.e. $A(R)$ is a near-ring with the defect $D'' = (0)$. By Proposition 1.3 $R$ is a near-ring with the defect $D = D' \oplus D'' = D'$.

Theorem 4.3. *Let $(R,S)$ be a near-ring with the defect $D \neq R$ and let $R$ be a direct sum of ideals which include $A(R)$, where $A(R)$ is finite. Then there exist the group $(G,+)$ and its normal subgroup $\Delta$ such that $R$ embeds in $E_\Delta(G)$.*

Proof. Let $R = A(R) \oplus B$. By Proposition 3 of [7], $A(R)$ embeds in some $E(G_1)$. Bu Lemma 2 of [7], $A(B) = (0)$. Since $D$ is a defect of $B$ (Proposition 4.2), it follows that $B$ embeds in $E_D(B)$ (Proposition 4.1). Thus, $R$ embeds in $\mathcal{R} = E(G_1) \oplus E_D(B)$, whereby multiplication on $\mathcal{R}$ is componentwise. Let $\mathcal{D}$ be a defect of the near-ring $E_D(B)$. Then, by Proposition 1.3 it follows that $\mathcal{R}$ is a near-ring with defect $\mathcal{D} \neq \mathcal{R}$, because the defect of $E(G_1)$ is zero. The nearring $\mathcal{R}$ contains identity $e = (e_1, e_2)$, where $e_1 \in E(G_1)$ and $e_2 \in E_D(B)$ are identity mappings, thus $A(\mathcal{R}) = (0)$. Hence by Proposition 4.1 $\mathcal{R}$ embeds in $E_D(\mathcal{R})$. Consequently, there exist the group $(G,+)$ and its normal subgroup $\Delta$ such that $R$ embeds in $E_\Delta(G)$.

Definition. Let $(R,+)$ be a direct sum of the subgroups $(A,+)$ and $(B,+)$. Let $(A,+,.)$ and $(B,+,.)$ be two subnear-rings of the nearring

**TABLE 1.**

The $\Delta$-endomorphisms of $(Z_4, +)$ for $\Delta=\{0, 2\}$.

The group $(E_\Delta(Z_4), +)$ and the semigroup $(E_\Delta(Z_4), \circ)$· $\Delta$-endomorphisms

| 0123 | + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_0=0000$ | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| $f_1=0123$ | 1 | 1 | 3 | 0 | 2 | 7 | 12 | 13 | 9 | 11 | 14 | 8 | 15 | 6 | 5 | 4 | 10 |
| $f_2=0321$ | 2 | 2 | 0 | 3 | 1 | 14 | 13 | 12 | 4 | 10 | 7 | 15 | 8 | 5 | 6 | 9 | 11 |
| $f_3=0202$ | 3 | 3 | 2 | 1 | 0 | 9 | 6 | 5 | 14 | 15 | 4 | 11 | 10 | 13 | 12 | 7 | 8 |
| $f_4=0103$ | 4 | 4 | 7 | 14 | 9 | 3 | 8 | 15 | 2 | 6 | 0 | 12 | 13 | 11 | 10 | 1 | 5 |
| $f_5=0121$ | 5 | 5 | 12 | 13 | 6 | 8 | 3 | 0 | 11 | 9 | 13 | 7 | 14 | 2 | 1 | 10 | 4 |
| $f_6=0323$ | 6 | 6 | 13 | 12 | 5 | 15 | 0 | 3 | 10 | 4 | 8 | 14 | 7 | 1 | 2 | 11 | 9 |
| $f_7=0222$ | 7 | 7 | 9 | 4 | 14 | 2 | 11 | 10 | 0 | 13 | 1 | 6 | 5 | 15 | 8 | 3 | 12 |
| $f_8=0220$ | 8 | 8 | 11 | 10 | 15 | 6 | 9 | 4 | 13 | 0 | 5 | 2 | 1 | 14 | 7 | 12 | 3 |
| $f_9=0301$ | 9 | 9 | 14 | 7 | 4 | 0 | 15 | 8 | 1 | 5 | 3 | 13 | 12 | 10 | 11 | 2 | 6 |
| $f_{10}=0101$ | 10 | 10 | 8 | 15 | 11 | 12 | 7 | 14 | 6 | 2 | 13 | 3 | 0 | 9 | 4 | 5 | 1 |
| $f_{11}=0303$ | 11 | 11 | 15 | 8 | 10 | 13 | 14 | 7 | 5 | 1 | 12 | 0 | 3 | 4 | 9 | 6 | 2 |
| $f_{12}=0200$ | 12 | 12 | 6 | 5 | 13 | 11 | 2 | 1 | 15 | 14 | 10 | 9 | 4 | 0 | 3 | 8 | 7 |
| $f_{13}=0002$ | 13 | 13 | 5 | 6 | 12 | 10 | 1 | 2 | 8 | 7 | 11 | 4 | 9 | 3 | 0 | 15 | 14 |
| $f_{14}=0020$ | 14 | 14 | 4 | 9 | 7 | 1 | 10 | 11 | 3 | 12 | 2 | 5 | 6 | 8 | 15 | 0 | 13 |
| $f_{15}=0022$ | 15 | 15 | 10 | 11 | 8 | 5 | 4 | 9 | 12 | 3 | 6 | 1 | 2 | 7 | 14 | 13 | 0 |

| $\circ$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 2 | 0 | 2 | 1 | 3 | 9 | 5 | 6 | 7 | 15 | 4 | 10 | 11 | 13 | 12 | 14 | 8 |
| 3 | 0 | 3 | 3 | 0 | 0 | 3 | 3 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 3 | 3 |
| 4 | 0 | 4 | 9 | 3 | 4 | 10 | 11 | 3 | 12 | 9 | 10 | 11 | 12 | 13 | 0 | 13 |
| 5 | 0 | 5 | 6 | 3 | 10 | 5 | 6 | 7 | 7 | 11 | 10 | 11 | 3 | 0 | 14 | 14 |
| 6 | 0 | 6 | 5 | 3 | 11 | 5 | 6 | 7 | 14 | 10 | 10 | 11 | 0 | 3 | 14 | 7 |
| 7 | 0 | 7 | 7 | 0 | 0 | 7 | 7 | 7 | 7 | 0 | 0 | 0 | 0 | 0 | 7 | 7 |
| 8 | 0 | 8 | 8 | 0 | 0 | 8 | 8 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 8 | 8 |
| 9 | 0 | 9 | 4 | 3 | 9 | 10 | 11 | 3 | 13 | 4 | 10 | 11 | 13 | 12 | 0 | 12 |
| 10 | 0 | 10 | 11 | 3 | 10 | 10 | 11 | 3 | 3 | 11 | 10 | 11 | 3 | 0 | 0 | 0 |
| 11 | 0 | 11 | 10 | 3 | 11 | 10 | 11 | 3 | 0 | 10 | 10 | 11 | 0 | 3 | 0 | 3 |
| 12 | 0 | 12 | 12 | 0 | 0 | 12 | 12 | 12 | 12 | 0 | 0 | 0 | 0 | 0 | 12 | 12 |
| 13 | 0 | 13 | 13 | 0 | 0 | 13 | 13 | 13 | 13 | 0 | 0 | 0 | 0 | 0 | 13 | 13 |
| 14 | 0 | 14 | 14 | 0 | 0 | 14 | 14 | 14 | 14 | 0 | 0 | 0 | 0 | 0 | 14 | 14 |
| 15 | 0 | 15 | 15 | 0 | 0 | 15 | 15 | 15 | 15 | 0 | 0 | 0 | 0 | 0 | 15 | 15 |

The near-ring $E_\Delta(Z_4)$ has the defect $\mathfrak{D}=\{f_0, f_3, f_{12}, f_{13}\}$.

TABLE 2.

The Δ-endomorphisms of $(Z_6, +)$ for which the subgroups $B_1 = \Delta = \{0, 3\}$ ane $B_2 = \{0, 2, 4\}$ are invariant. The group $(E_\Delta(Z_6), +)$.

Δ-endomorphisms

| 012345 | + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $_0 = 000000$ | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| $f_1 = 012345$ | 1 | 1 | 2 | 3 | 4 | 5 | 0 | 10 | 11 | 7 | 8 | 9 | 6 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 14 | 13 | 22 | 23 | 12 |
| $f_2 = 024024$ | 2 | 2 | 3 | 4 | 5 | 0 | 1 | 9 | 6 | 11 | 7 | 8 | 10 | 18 | 19 | 20 | 21 | 14 | 13 | 22 | 17 | 16 | 23 | 12 | 15 |
| $f_3 = 030303$ | 3 | 3 | 4 | 5 | 0 | 1 | 2 | 8 | 10 | 6 | 11 | 7 | 9 | 21 | 14 | 13 | 22 | 17 | 16 | 23 | 20 | 19 | 12 | 15 | 18 |
| $f_4 = 042042$ | 4 | 4 | 5 | 0 | 1 | 2 | 3 | 7 | 9 | 10 | 6 | 11 | 8 | 22 | 17 | 16 | 23 | 20 | 19 | 12 | 13 | 14 | 15 | 18 | 21 |
| $f_5 = 054321$ | 5 | 5 | 0 | 1 | 2 | 3 | 4 | 11 | 8 | 9 | 10 | 6 | 7 | 23 | 20 | 19 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 21 | 22 |
| $f_6 = 012045$ | 6 | 6 | 10 | 9 | 8 | 7 | 11 | 2 | 0 | 5 | 4 | 3 | 1 | 16 | 15 | 22 | 19 | 18 | 23 | 14 | 21 | 12 | 17 | 20 | 13 |
| $f_7 = 054021$ | 7 | 7 | 11 | 6 | 10 | 9 | 8 | 0 | 4 | 3 | 2 | 1 | 5 | 20 | 23 | 18 | 13 | 12 | 21 | 16 | 15 | 22 | 19 | 14 | 17 |
| $f_8 = 042342$ | 8 | 8 | 7 | 11 | 6 | 10 | 9 | 5 | 3 | 2 | 1 | 0 | 4 | 17 | 22 | 15 | 20 | 23 | 18 | 13 | 12 | 21 | 16 | 19 | 14 |
| $f_9 = 030003$ | 9 | 9 | 8 | 7 | 11 | 6 | 10 | 4 | 2 | 1 | 0 | 5 | 3 | 14 | 21 | 12 | 17 | 22 | 15 | 20 | 23 | 18 | 13 | 16 | 19 |
| $f_{10} = 024324$ | 10 | 10 | 9 | 8 | 7 | 11 | 6 | 3 | 1 | 0 | 5 | 4 | 2 | 19 | 18 | 23 | 14 | 21 | 12 | 17 | 22 | 15 | 20 | 13 | 16 |
| $f_{11} = 000300$ | 11 | 11 | 6 | 10 | 9 | 8 | 7 | 1 | 5 | 4 | 3 | 2 | 0 | 13 | 12 | 21 | 16 | 15 | 22 | 19 | 18 | 23 | 14 | 17 | 20 |
| $f_{12} = 000003$ | 12 | 12 | 15 | 18 | 21 | 22 | 23 | 16 | 20 | 17 | 14 | 19 | 13 | 0 | 11 | 9 | 1 | 6 | 8 | 2 | 10 | 7 | 3 | 4 | 5 |
| $f_{13} = 000303$ | 13 | 13 | 16 | 19 | 14 | 17 | 20 | 15 | 23 | 22 | 21 | 18 | 12 | 11 | 0 | 3 | 6 | 1 | 4 | 10 | 2 | 5 | 9 | 8 | 7 |
| $f_{14} = 030000$ | 14 | 14 | 17 | 20 | 13 | 16 | 19 | 22 | 18 | 15 | 12 | 23 | 21 | 9 | 3 | 0 | 8 | 4 | 1 | 7 | 5 | 2 | 11 | 6 | 10 |
| $f_{15} = 012342$ | 15 | 15 | 18 | 21 | 22 | 23 | 12 | 19 | 13 | 20 | 17 | 14 | 16 | 1 | 6 | 8 | 2 | 10 | 7 | 3 | 9 | 11 | 4 | 5 | 0 |
| $f_{16} = 012042$ | 16 | 16 | 19 | 14 | 17 | 20 | 13 | 18 | 12 | 23 | 22 | 21 | 15 | 6 | 1 | 4 | 10 | 2 | 5 | 9 | 3 | 0 | 8 | 7 | 11 |
| $f_{17} = 042345$ | 17 | 17 | 20 | 13 | 16 | 19 | 14 | 23 | 21 | 18 | 15 | 12 | 22 | 8 | 4 | 1 | 7 | 5 | 2 | 11 | 0 | 3 | 6 | 10 | 9 |
| $f_{18} = 024021$ | 18 | 18 | 21 | 22 | 23 | 12 | 15 | 14 | 16 | 13 | 20 | 17 | 29 | 2 | 10 | 7 | 3 | 9 | 11 | 4 | 8 | 6 | 5 | 0 | 1 |
| $f_{19} = 024321$ | 19 | 10 | 14 | 17 | 20 | 13 | 16 | 21 | 15 | 12 | 23 | 22 | 18 | 10 | 2 | 5 | 9 | 3 | 0 | 8 | 4 | 1 | 7 | 11 | 6 |
| $f_{20} = 054024$ | 20 | 29 | 13 | 16 | 19 | 14 | 17 | 12 | 22 | 21 | 18 | 15 | 23 | 7 | 5 | 2 | 11 | 0 | 3 | 6 | 1 | 4 | 10 | 9 | 8 |
| $f_{21} = 030300$ | 21 | 21 | 22 | 23 | 12 | 15 | 18 | 17 | 19 | 16 | 13 | 20 | 14 | 3 | 9 | 11 | 4 | 8 | 6 | 5 | 7 | 10 | 0 | 1 | 2 |
| $f_{22} = 042045$ | 22 | 22 | 23 | 12 | 15 | 18 | 21 | 20 | 14 | 19 | 16 | 13 | 17 | 4 | 8 | 6 | 5 | 7 | 10 | 0 | 11 | 9 | 1 | 2 | 3 |
| $f_{23} = 054324$ | 23 | 23 | 12 | 15 | 18 | 21 | 22 | 13 | 17 | 14 | 19 | 16 | 20 | 5 | 7 | 10 | 0 | 11 | 9 | 1 | 6 | 8 | 2 | 3 | 4 |

The semigroup $(E_\Delta(Z_6), \circ)$. The near-ring $E_\Delta(Z_6)$ has the defect

$$\mathcal{D}=\{f_0, f_9, f_{12}, f_{14}\}.$$

| ∘ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 | 2 | 4 | 2 | 0 | 4 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 4 | 4 | 4 | 0 | 2 | 4 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 0 | 3 | 0 | 3 | 3 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 3 | 0 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 | 4 | 2 | 4 | 0 | 2 | 0 | 0 | 0 | 0 | 4 | 4 | 4 | 2 | 2 | 2 | 0 | 4 | 2 |
| 5 | 0 | 5 | 4 | 3 | 2 | 1 | 7 | 6 | 10 | 9 | 8 | 11 | 14 | 21 | 12 | 19 | 18 | 23 | 16 | 15 | 22 | 13 | 20 | 17 |
| 6 | 0 | 6 | 2 | 9 | 4 | 7 | 6 | 7 | 4 | 9 | 2 | 0 | 0 | 12 | 14 | 16 | 16 | 22 | 18 | 18 | 20 | 14 | 22 | 20 |
| 7 | 0 | 7 | 4 | 9 | 2 | 6 | 7 | 6 | 2 | 9 | 4 | 0 | 14 | 14 | 12 | 18 | 18 | 20 | 16 | 16 | 22 | 12 | 20 | 22 |
| 8 | 0 | 8 | 2 | 11 | 4 | 10 | 4 | 2 | 8 | 0 | 10 | 11 | 0 | 11 | 0 | 8 | 4 | 8 | 2 | 10 | 2 | 11 | 4 | 10 |
| 9 | 0 | 9 | 0 | 9 | 0 | 9 | 0 | 0 | 9 | 0 | 9 | 9 | 0 | 9 | 0 | 9 | 0 | 9 | 0 | 9 | 0 | 9 | 0 | 9 |
| 10 | 0 | 10 | 4 | 11 | 2 | 8 | 2 | 4 | 10 | 0 | 8 | 11 | 0 | 11 | 0 | 10 | 2 | 10 | 4 | 8 | 4 | 11 | 2 | 8 |
| 11 | 0 | 11 | 0 | 11 | 0 | 11 | 0 | 0 | 11 | 0 | 11 | 11 | 0 | 11 | 0 | 11 | 0 | 11 | 0 | 11 | 0 | 11 | 0 | 11 |
| 12 | 0 | 12 | 0 | 0 | 0 | 12 | 0 | 12 | 12 | 0 | 12 | 12 | 0 | 12 | 0 | 12 | 0 | 12 | 0 | 12 | 0 | 12 | 0 | 12 |
| 13 | 0 | 13 | 0 | 13 | 0 | 13 | 0 | 0 | 13 | 0 | 13 | 13 | 0 | 13 | 0 | 13 | 0 | 13 | 0 | 13 | 0 | 13 | 0 | 13 |
| 14 | 0 | 14 | 0 | 14 | 0 | 14 | 0 | 0 | 14 | 0 | 14 | 14 | 0 | 14 | 0 | 14 | 0 | 14 | 0 | 14 | 0 | 14 | 0 | 14 |
| 15 | 0 | 15 | 2 | 3 | 4 | 23 | 16 | 20 | 8 | 14 | 10 | 11 | 0 | 11 | 14 | 15 | 16 | 18 | 2 | 10 | 20 | 21 | 4 | 23 |
| 16 | 0 | 16 | 2 | 14 | 4 | 20 | 16 | 20 | 4 | 14 | 2 | 0 | 0 | 0 | 14 | 16 | 16 | 4 | 2 | 2 | 20 | 14 | 4 | 20 |
| 17 | 0 | 17 | 2 | 13 | 4 | 19 | 22 | 18 | 8 | 12 | 10 | 11 | 12 | 13 | 0 | 8 | 4 | 17 | 18 | 19 | 2 | 11 | 22 | 10 |
| 18 | 0 | 18 | 4 | 12 | 2 | 22 | 18 | 22 | 2 | 12 | 4 | 0 | 0 | 0 | 12 | 18 | 18 | 18 | 4 | 4 | 22 | 12 | 2 | 22 |
| 19 | 0 | 19 | 4 | 13 | 2 | 17 | 18 | 22 | 10 | 12 | 8 | 11 | 0 | 11 | 12 | 19 | 18 | 10 | 4 | 8 | 22 | 13 | 2 | 17 |
| 20 | 0 | 20 | 4 | 14 | 2 | 16 | 20 | 16 | 2 | 14 | 4 | 0 | 14 | 14 | 0 | 2 | 2 | 20 | 16 | 16 | 4 | 0 | 20 | 4 |
| 21 | 0 | 21 | 0 | 21 | 0 | 21 | 0 | 0 | 21 | 0 | 21 | 21 | 0 | 21 | 0 | 21 | 0 | 21 | 0 | 21 | 0 | 21 | 0 | 21 |
| 22 | 0 | 22 | 2 | 12 | 4 | 18 | 22 | 18 | 4 | 12 | 2 | 0 | 12 | 12 | 0 | 4 | 4 | 22 | 18 | 18 | 2 | 0 | 22 | 2 |
| 23 | 0 | 23 | 4 | 21 | 2 | 15 | 20 | 16 | 10 | 14 | 8 | 11 | 14 | 21 | 0 | 10 | 2 | 23 | 16 | 15 | 4 | 11 | 20 | 8 |

$(R, +, .)$ with the defect $D$. A multiplication on $R$ is $D$-componentwise if for all $a,\ a' \in A$ and $b,\ b' \in B$ there exists $d \in D$ such that $(a+b)(a'+b') = aa' + bb + d$. We say that $R$ is a $D$-direct sum of thee subnear-rings $A$ and $B$.

Let $E_\Delta(G)$ be a $\Delta$-endomorphism near-ring with the defect $\mathcal{D}$. For some idempotent $e \in E_\Delta(G)$ let $\mathcal{A}$ be the subgroup of $(E_\Delta(G), +)$ generated by $\{s - es\colon s \in End_\Delta(G)\}$ and $\mathcal{M}$ be the subgroup of $(E_\Delta(G), +)$ generated by $\{es\colon s \in End_\Delta(G)\}$.

THEOREM 4.4. *Let $G = B \oplus C$ be a direct sum of $E_\Delta$-invariant subgroups $B$ and $C$, where $B$ is summand and $\Delta$ is a subset of one of the summands. If $e$ is the projection map $e\colon G \to B$ and $\mathcal{AM} \subseteq \mathcal{D}$, then $E_\Delta(G)$ is the $\mathcal{D}$-direct sum of the subnear-rings $\mathcal{A}$ and $\mathcal{M}$, where $\mathcal{D}$ is the defect of $E_\Delta(G)$.*

PROOF. The projection map $e\colon G \to B$ is an endomorphism of $(G, +)$. The idempotent $e \in End(G)$ is a right identity for $\mathcal{M}$. Hence, $\mathcal{M}$ is a subnear-ring of $E_\Delta(G)$. Also, by Corollary 2.3 of [11] it follows that $\mathcal{A}$ is an ideal of $E_\Delta(G)$. Because $B$ and $C$ commute elementwise and $B$ is $E_\Delta$-invariant abelian summand, it follows that the decomposition $E_\Delta(G) = \mathcal{A} + \mathcal{M}$ has $\mathcal{M}$ in the additive center of $E_\Delta(G)$, i.e. semidirect sum $\mathcal{A} + \mathcal{M}$ is direct.

We shall now prove that the multiplication on $E_\Delta(G)$ is $\mathcal{D}$-componentwise. Let $a, a' \in \mathcal{A}$ and $m, m' \in \mathcal{M}$, where $a' = s' - es'$, $m = et$, $m' = et'(s', t, t' \in End_\Delta(G))$. Then

$$\begin{aligned}
(a+m)(a'+m') &= (a+m)(s'-es') + (a+m)et' \\
&= (a+m)s' - (a+m)es' + (a+m)et' = \\
&= as' + ms' + \delta_1 - (aes' + mes' + \delta_2) + aet' + met' + \delta_3 = \\
&= as' - aes' + aet' + ms' - mes' + met' + \delta = \\
&= aa' + am' + ma' + mm' + \delta \\
&= aa' + mm' + \delta', \ \ (\delta_1, \delta_2, \delta_3, \delta, \delta' \in \mathcal{D})
\end{aligned}$$

because $ma' = et(s' - es') = ets' - etes' = 0$ and $\mathcal{AM} \subseteq \mathcal{D}$.

For example, if for an idempotent of the near-ring $E_\Delta(Z_6)$ with the defect $\mathcal{D} = \{f_0, f_9, f_{12}, f_{14}\}$ (table 2) we take the map $e = f_3\colon G \to B = \{0, 3\}$ then, $E_\Delta(Z_6)$ is a $\mathcal{D}$-direct sum of the subnear-rings

$$\mathcal{A} = \{f_0, f_2, f_4, f_6, f_7, f_9, f_{11}, f_{12}, f_{13}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$$

and $\mathcal{M} = \{f_0, f_3\}$

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