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M-CONVEXITY AND BEST APPROXIMATION

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Abstract. The notion of M-convexity is introduced in Metric Spaces. The relations betwen M-convex, strictly M-convex and uniformly M-covex metric spaces are studied. The Best approximation properties for M-convex subsets of metric spaces are considered and many new results derived.

Introduction. The problem of best approximation has been extensively studied in normed linear spaces. The study of similar problems in metric linear spaces was initiated by G. Albinus [1, 2, 3] and I. Singer [12]. In metric linear spaces the proximinality of finite dimensional linear subspaces has been studied by K. Iseki [7] and in Frechet spaces, similar problems have been considered by V.N. Nikolski [10]. The consideration of best approximation problems in normed linear spaces was made by B.A. Hirschfeld [6] and A.M. Flomin [5]. In metric spaces many of the results of metric linear spaces were extended by I. Singer [12] and other. In this paper we have studied some such problems in a new kind of metric space which admits extensions of many results true in metric linear spaces and normed linear spaces.

In Section 1, we have defined M-convexity for metric spaces. The idea is essentially due to K. Menger who has survived in the prefix of the concept. Strict M-convexity and uniform M-convexity have been defined then and the relations among these spaces have been studied. In Sectton 2 M-convexity is defined for subsets and Chebyshev and proximinal properties are studied there.

Definition 1.1. A metrix space (X, d) is said to be M-convex if for every x, y in X, $x \neq y$, there exists a z in X different from x and y such that

$$d(x,y) = d(x,z) + d(z,y).$$

One can immediately see that

Proposition 1.2. Every normed linear linear space is an M-convex metric space.

We now give two exmples to illustrate the fact that not every metric space is M-convex and also the fact not every M-convex metrix space is a normed linear space.

EXAMPLE 1.3. Let K be a non-convex closed subset of R^n equipped with the relative topology. Then it is easy to see that K is a metric space which is not M-convex.

EXAMPLE 1.4. Let U denote the unit ball of R^2 i.e.

$$U = \{(x,y) \in R^2; x^2 + y^2 \leq 1\}$$

Then, if U is equipped with the usual Euclidean metric, then U becomes an M-convex metric space which is not a normed linear space.

DEFINITION 1.5. A metric space (X,d) is said to be strictly M-convex if for every x, y, t in X, all different and $r > 0$, there exists a z in X different from x, y and t such that

$$d(x,y) = d(x,z) + d(z,y) \tag{1}$$

$$d(x,t) \leq r, d(y,t) \leq r \text{ imply } d(z,t) < r \tag{2}$$

An example of an M-convex metric space which is not strictly M-convex is the following

EXAMPLE 1.6. Consider the metrix space (X,d) where d is defined as

$$d(x,y) = \max\{|x_1 - y_1|, \quad |x_2 - y_2|\}$$
$$\text{where } x = (x_1, x_2), \quad y = (y_1, y_2)$$
$$\text{and} \qquad X = \{x,y) \in R^2; \quad x > 0, \ y > 0\} \cup (0,0).$$

Then it is easy to see th that (X,d) is M-convex. To see that it is not strictly M-convex, consider

$$x = (1/3, 1/3) \qquad y = (2/3, 2/3)$$
$$t = (0,0) \qquad r = 2/3.$$

and check that does not exist any z in X satisfying both the requirements of the definition.

DEFINITION 1.7. A metrix space (X,d) is said to be uniformly M-*convex* if for every pair of positive numbers ε and r, there corresponds a positive number δ such that for every triplet x, y, t in X all different and satisfying $d(x,y) \geq \varepsilon$, $d(x,t) < r + \delta$, $d(y,t) < r + \delta$ there exists a z in X with the properties

$$d(x,y) = d(x,z) + d(z,y) \tag{1}$$

$$d(z,t) < r. \tag{2}$$

As an example of a uniformly M-convex metric space we site the following example.

EXAMPLE 1.8. Consider (M,d) with metric d defined as $d(x,y)=|x-y|$.

To see that it is uniformly M-convex, it is enough to check $d(x,y)\geq\varepsilon$, $d(x,0)<r'$, $d(y,0)<r'$, there exists a z in X defferent from x, y, t such that

$$d(x,y)=d(x,z)+d(z,y) \tag{1}$$

$$d(z,0)<r. \tag{2}$$

This can be verified easily.

One can readily see now.

PROPOSITION 1.9. Every uniformly M-convex metric space is strictly M-convex and not conversely.

DEFINITION 1.10. A metric space (X,d) is said to be *totally complete* if every bounded closed subset of X is compact.

We can now prove the following theorem.

THEOREM 1.11. *Every totaly complete strictly M-convex metrix space is unifomly M-convex.*

PROOF. Let (X,d) be a totally complete M-convex metric space. Equip $X\times X$ with a metric ρ defined as

$$\rho((x_1,y_1),\ (x_2,y_2))=\{d_2(x_1,x_2)+d^2(y_1,y_2)\}^{\frac{1}{2}}$$

clearly, $(X\times X,\rho)$ is totally complete.

Then $S_t=\{(x,y)\in X\times X;\ d(x,t)\leq r\}$ is a closed and bounded subset of $X\times X$ and hence compact for every $t\in X$.

Define $\Phi_t\colon S_t\to R$ as

$$\Phi_t(x,y)=r-d(z,t)\ \text{ where }\ d(x,z)+d(z,y)=(x,y).$$

Then Φ_t is continuous and positive on S_t and therefore there exists $\delta>0$ such that

$$r-d(z,t)\geq\delta\ \text{ for all }\ t\ \text{ in }\ X$$
$$\text{i.e. }\ d(z,t)\leq r-\delta<r.$$

Hence the result.

2. We now define M-convex subsets of a metrix space.

DEFINITION 2.1. A subset G of a metric space (X,d) is said to be M-convex if for every $x,y\in G$, $x\neq y$, there exists a z in G such that $d(x,z)+d(z,y)=d(x,y)$.

We remark here that there are metric spaces no subset of which is M-convex. As an example of such a metrix space we refer to the following.

EXAMPLE 2.2. Consider $X = R$ with the metrix d defined as

$$d(x,y) = \frac{|x-y|}{1+|x-y|}.$$

It is worthy to note at this stage that there is no relation as such betwen convexity and M-convexity in metric linear spaces. The example 2.2 provides an illustration to the fact that in metric linear spaces there are convex sets which are not M-convex while example 2.3 below proves the other direction.

EXAMPLE 2.3. Consider the metric linear space (X,d) where $X = R^2$ and d is defined as

$$d(x,y) = \max\{|x_1 - y_1|, \ |x_2 - y_2|\} \text{ where } \begin{array}{l} x - (x_1, y_2) \\ y = (y_1, y_2). \end{array}$$

Consider the set

$$G = \{(z_1, z_2); \quad 0 \le z_1 \le 2, \ z_2 = 0\} \cup \{(z_1, z_2); \quad z_1 = 2, \ 0 \le z_2 \le 1\}$$

Then it is easy to check that G is M-convex but not convex.

We however remark that in normed linear spaces this is not true.

DEFINITION 2.4. A subset G of a metric space (X,d) is said to be *promiminal* if for every x in X there exists at least one ξ in G, called the best approximating element of x in G such that

$$d(x,\xi) = d(x,G) \equiv \inf_{z \in G} d(x,z)$$

The set G is said to be *Chebyshev* if to every $x \in X$ there exists exactly one $\xi \in G$ such that

$$d(x,\xi) = d(x,G).$$

We shall denote by $\Pi_{G(x)}$ the set of best approximating elements of x in G.

Thus we can define the setvalued map Π_G from X into subsets of G as

$$\Pi_G(x) = \{z \in G; \quad d(x,z) = d(x,G)\}$$

We can define another real-valued function e_G on X as $e_G(x) = d(x,G)$.

Clearly e_G is uniformly continuous. Regarding continuity of $\Pi_{G(x)}$, it is known that Π_G is continuous at every point of G if G is Chebyshev.

We can prove the following theorems now.

THEOREM 2.5. *If (X,d) is an M-convex metric space and G is a Chebyshev subset of X then if z corresponds the M-convex element of x and $\Pi_{G(x)}$*

$$\Pi_G(z) = \Pi_G(x).$$

PROOF. By the definition of z, we have

$$d(x,z) + d(z,\Pi_G(x)) = d(x,\Pi_G(x)).$$

Now, if $\xi \in G$, then

$$\begin{aligned} &d(z,\xi) \geq d(x,\xi) - d(x,z) \\ &d(x,\Pi_G(x)) - d(x,z) \\ &= d(z,\Pi_G(x)) \end{aligned}$$

This implies that $\Pi_G(x)$ is a best approximating element of z. Since G is Chebyshev, $\Pi_G(x) = \Pi_G(z)$.

THEOREM 2.6. *If (X,d) is a metric space, G is a subset of X and $y_0 \in G$, then $\Pi_G^{-1}(y_0)$ is closed.*

Further, if $x_0 \in \Pi_G^1(y_0)$ and $d(x,z) + d(z,y_0) = d(x_0,y_0)$ for some $z \in X$, then $z \in \Pi_G^{-1}(y_0)$.

PROOF. By definition

$$\begin{aligned} \Pi_G^{-1}(y_0) &= \{x \in X; \quad d(x,y_0) = d(x,G)\} \\ &= \bigcap_{y \in G} \{x \in X; \quad d(x,y_0) \leq d(x,y)\} \end{aligned}$$

By the continuity of the metric d, the first part of the result is then obvious. Now since $x_0 \in \Pi_G^{-1}(y_0)$, we have

$$d(x_0,y_0) \leq d(x_0,y) \text{ for all } y \in G.$$

Since z satisfies $d(x_0,z) + d(z,y_0) = d(x_0,y_0)$, we write

$$\begin{aligned} d(x,y_0) &= d(x_0,y_0) - d(x_0,z) \\ &\leq d(x_0,y_0) \leq d(x_0,y) \text{ for all } y \in G \end{aligned}$$

$\therefore z \in \Pi_G^{-1}(y_0)$.

COROLLARY 2.7. If G is Chebyshev in X, then $\Pi_G^{-1}(\Pi_G(x))$ is closed for every x in X.

In general, proximinal sets or Chebyshev sets are neither convex nor M-convex. L.N.H. Bunt [4] and T.S. Motzkin have given conditions under which every Chebyshev set is convex. These conditions are however sufficient but not necessary. One such result is the following.

THEOREM. *In a finite dimensional smooth Banach Space, every Chebyshev set is convex and hence M-convex. But the problem whether the result is true for infinite dimensional Banach Spaces remains still open. Another interesting open problem is the following.*

Whether in a Hilbert Space, every Chebyshev set is convex? Under the present context, we can ask whether in a Hilbert space, every Chebyshev set is M-convex.

Before we prove our next theorem we need the following definition.

DEFINITION 2.8. In a metric space (X, d), a Menger set denoted as $M_{<x,y>}$ for a pair of distinct points x, y is defined as the set of elements z in X such that

$$d(x, z) + d(z, y) = d(x, y)$$
$$\text{i.e, } M_{<x,y>} = \{z \in X; \quad d(x, z) + d(z, y) = d(x, y)\}.$$

One can immediately see that Menger sets can be emptysets, singleton sets or arbitrarily large sets. For example, we recall that (R, d) with $d(x, y) = \frac{|x-y|}{1+|x-y|}$ has empty Menger sets for every pair of points of x, y while (R, d') with $d'(x, y) = |x-y|$ is such that the Menger set for every pair of distinct points is uncountable.

One can immediattely note

PROPOSITION 2.9. Every Menger set is closed.

DEFINITION 2.10. If a metric space has only singleton Menger sets for every pair of distinct elements, then it will be called Mengerian. We can prove the following theorem now.

THEOREM 2.11. *Every* M-*convex proximinal set in a strictly* M-*convex Mengerian metric space is Chebyshev.*

PROOF. Suppose G is an M-convex promiminal set in the strictly M-convex Mengerian metric space (X, d).

If possible, for some $x_0 \in X$, let $y_1, y_2 \in G$ be two best approximating elements i.e. $\Pi_G(x_0) = \{y_1, y_2\}$.

Then $d(x_0, y_1) = d(x_0, y) = \inf_{\xi \in G} d(x, \xi) = r$ say.

Since X is strictly M-convex, there exists $z \in X$, $x \neq z \neq y$ such that $d(x, z) < r$

$$\text{and } d(y_1, z) + d(z, y_2) = d(y_1, y_2).$$

But since G is M-convex and X is Mengerian, $z \in G$.

This contradicts the definition of r. Hence the proof.

DEFINITION 2.12. A set G in a metric space (X, d) is said to be *approximatively compact* if for every sequence y_n in G with $\lim_n d(x, y_n) = d(x, G)$, there exists a subsequence y_{n_k} coverging to an element of G.

THEOREM 2.13. *In an unifomly M-convex Mengrian metric space every complete M-convex set is approximatively compact.*

PROOF. Let G be an M-convex complete set in uniformly M-convex Mengerian metric space (X, d).

Let y_n be a sequence in G satisfying

$$\lim d(x, y_n) = d(x, G) = r \text{ (say)}$$

Let $\varepsilon > 0$ be arbitrary.

Since X is uniformly M-convex, we can find a $\delta > 0$ satisflying some inequality relations.

Since $\lim_n d(x, y_n) = r$, we can choose a positive integer N such that

$$d(x, y_n) > r + \delta \text{ whenever } n \geq N.$$

Let $n, m \geq N$. Then by the inequality relations, we get

$$\begin{aligned} d(x, y_n) &< r + \delta \\ d(x, y_m) &< r + \delta. \end{aligned}$$

If possible let $d(y_n, y_m) \geq \varepsilon$.

Then these imply that there exists a $y \in X$ such that

$$d(y_n, y_0) + d(y_0, y_m) = d(y_n, y_m)$$

and

$$d(x, y_0) < r.$$

Since X is M-convex and Mengerian, $y \in G$ and thus we arrive at a contradiction that $r = d(x, G)$.

Therefore $d(y_n, y_m) < \varepsilon$ for $m, n \geq N$.

i.e. y_n is a Cauchy sequence.

By the completeness of G, the result follows then.

DEFINITION 2.14. A metric space (X, d) is said to have *P-property* if for a fixed p in X, every sequence y_n in a M-convex set G of X satisfying $\lim_n d(p, y_n) = d(p, G)$ has a Cauchy subsequence.

The following theorem can now be proved.

THEOREM 2.15. *A complete M-convex subset G of a metric space (X,d) having P-property is Chebyshev.*

PROOF. Let $p \in X$ and $r = d(x,G)$.

So there exists a sequence y_n in G such that

$$\lim_n d(p, y_n) = r$$

By P-property, y_n has a Chauchy subsequence y_{n_k} in G. Since G is complete, y_n converges to some $y \in G$.

By continuity of the metric, we then get

$$d(p,y) = r.$$

If possible now let $y_1, y_2 \in G$ be such that

$$d(p,y_1) = d(p,y_2) = r.$$

Define a sequence z_n as follows

$$\begin{aligned} z_n &= y_1 \text{ if } n \text{ is even} \\ &= y_2 \text{ if } n \text{ is odd.} \end{aligned}$$

Then $\lim_n d(p,z) = d(p,z_1) = r = d(p,z_2)$.

By P-proporety z_n has a Cauchy subsequence z_{n_k} i.e., for given $\xi > 0$, there exists a positive integer N such that $n_k, m_k \geq N$ implies $d(z_{n_k}, z_{m_k}) < \xi$.

Since ε is arbitrary, $y_1 = y_2$.

This proves that G is Chebyshev.

THEOREM 2.16. *Every uniformly M-convex Mengerian metric space has P-property.*

PROOF. Embodied in Theorem 2.14.

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