PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 28 (42), 1980, 5–11

# CERTAIN CONVEXITY THEOREMS FOR UNIVALENT ANALYTIC FUNCTIONS

*S.K. Bajpai and S.P. Dwivedi*

**1.** Let $m$ and $M$ be arbitrary fixed real number which satisfy the relations $(m, M) \in R$ where $R = \big\{(m, M) \mid m > \frac{1}{2}, (m-1) < M < m\big\}$. Also, let $P$ denote the class of functions $F(z) = 1 + c_0 z + c_1 z^2 + \cdots$ which are regular and satisfy $\Re\{F(z)\} > \alpha$; $0 \leq \alpha < 1$ and $|F(z) - m| < M$. Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular in the unit disc $D = \{z \mid |z| < 1\}$ and write $G(z) = zf'(z)/f(z)$ and $H(z) = 1 + zf'(z)/f(z)$. Then we denote the class of functions $G(z)$ for which $G(z) \in P$ by $S(m, M)$ while the class of functions $H(z) \in P$ by $K(m, M)$. We further assume that

$$a = \frac{M^2 - m^2 + m}{M} \quad \text{and} \quad b = \frac{m-1}{M}.$$

Then, it follows that

$$f \in S(m, M) \Leftrightarrow \frac{zf'(z)}{f(z)} = \frac{1 + aw(z)}{1 - bw(z)}$$

where $w(z)$ is regular in $D$ and satisfy $w(0) = 0$, $|w(z)| < 1$. Similarly,

$$f \in K(m, M) \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} = \frac{1 + aW(z)}{1 - bW(z)}$$

where $W(0) = 0$, $|W(z)| < 1$ and is regular in $D$. If we write $a = \{\alpha - 2N\alpha + N\}/N$ and $b = (N-1)/N$ and make $N \to \infty$ then, it is equivalent to say that $b \to 1$ and $a \to 1 - 2\alpha$. In this case define

$$S^*(\alpha) = \lim_{\substack{a \to 1-2\alpha \\ b \to 0}} S(m, M); \qquad 0 \leq \alpha < 1$$

and

$$K(\alpha) = \lim_{\substack{a \to 1-2\alpha \\ b \to 0}} K(m, M); \qquad 0 \leq \alpha < 1.$$

The functions in $S^*(\alpha)$ and $K(\alpha)$ are usual functions of starlike univalent functions of order $\alpha$ and convex functions of order $\alpha$.

Also, if we let $S_0$ be the class of regular functions $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$ in $D_0 = \{z \mid |0 < |z| < 1\}$ and $Q$ denote the class of functions $F$ regular and satisfy $|F(z) + m| < M$, then define the class of functions:

$$\Gamma(m, M) = \left\{ f \mid f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \in S_0 \text{ and } zf'(z)/f(z) \in Q \right\}$$

and

$$\sum(m, M) = \left\{ f \mid f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \in S_0 \text{ and } 1 + zf'(z)/f(z) \in Q \right\}.$$

As before, we have

$$f \in \Gamma(m, M) \Leftrightarrow \frac{zf'(z)}{f(z)} = -\frac{1 + aw_1(z)}{1 - bw_1(z)}$$

and

$$f \in \sum(m, M) \Leftrightarrow 1 + \frac{zf''(z)}{f'(z)} = -\frac{1 + aw_2(z)}{1 - bw_2(z)}$$

where $w_i$; $i = 1, 2$ are regular in $D$ and satisfy $w_i(0) = 0$, $|w_i(z)| < 1$. Also it follows that

$$\Gamma^*(\alpha) = \lim_{\substack{a \to 1-2\alpha \\ b \to 0}} \Gamma(m, M), \qquad 0 \le \alpha < 1,$$

and

$$\sum(\alpha) = \lim_{\substack{a \to 1-2\alpha \\ b \to 0}} \sum(m, M); \qquad 0 \le \alpha < 1.$$

Then $\Gamma^*(\alpha)$ and $\sum(\alpha)$ denote the usual class of starlike and convex functions in $D_0$. In this paper, we shall prove the following theorems which in particular include the results proved in [1–3] or else obtained as a lim t $a \to 1 - 2\alpha$ and $b \to 0$.

**2.** We have:

THEOREM 1: *Let $f \in S(m, M)$ and*

$$F(z) = \left(\frac{c+1}{z^c}\right) \int_0^z t^{c-1} f(t)\,dt, \quad c > -\frac{1-a}{1+b} \tag{2.1}$$

*where $a$, $b$ are defined by*

$$a = \frac{M^2 - m^2 + m}{M} \text{ and } b = \frac{m-1}{b}; \quad (m, M) \in R \tag{2.2}$$

*and $r(a,b)$ be the unique positive root of the equation*

$$
\begin{aligned}
(2.3)\qquad & (a+2b+d) - 2(ad+bd+b+d)r - \{2(b^2-d^2) + (a+d) + 2b(1-d^2) \\
& \qquad - d(ad+b^2)\}r^2 - 2d\{(a+b) + b(b+d)r^3\} \\
& \qquad - d(ad+2bd+b^2)r^4 = 0.
\end{aligned}
$$

*then, $f(z)$ is starlike of order $\beta$ for $|z| < r_0$, where $r_0$ is the smallest positive root of the equation*

$$(2.4)\qquad (1-\beta) - \{\beta(b-d) + a + b + 2d\}r + d(a+b\beta)r^2 = 0$$

*if $r_0 \le r(a-b)$, otherwise $r_0$ is the smallest positive root of the equation*

$$
\begin{aligned}
(2.5)\qquad 0 = & \sqrt{(1-d)\{(1-d)+(1+d)x\}\{(1+2a+4b+b^2)+(1-b^2)x\}}+ \\
& + (E-1+bd) - (1+bd)x
\end{aligned}
$$

*where*

$$(2.6)\qquad x = \frac{1+r^2}{1-r^2}, \quad E = -\beta(b+d) + 2d - (a+b) \quad \text{and} \quad d = \frac{a-bc}{c+1}.$$

*This result is sharp.*

Proof. Since $F \in S(m,M)$ there exists a regular function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ and

$$(2.7)\qquad \frac{zF'(z)}{F(z)} = \frac{1+aw(z)}{1+bw(z)}.$$

From (2.7) and (2.1) we get

$$(2.8)\qquad \frac{f(z)}{F(z)} = \frac{1 + \frac{a-bc}{c+1}w(z)}{1-bw(z)} = \frac{1+dw(z)}{1-bw(z)}.$$

Differentiating (2.8) logarithmically with respect to $z$ and using (2.7), we get,

$$
\begin{aligned}
(2.9)\qquad & \Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \ge -\beta + \Re\left\{\frac{1+aw(z)}{1-bw(z)}\right\} + \\
& +(b+d)\Re\left\{\frac{w(z)}{(1-bw(z))(1+dw(z))}\right\} - \frac{(b+d)(r^2-|w(z)|^2)}{(1-r^2)|1-bw(z)|\,|1+dw(z)|}.
\end{aligned}
$$

Here we have used the well known inequality

$$|zw'(z) - w(z)| \le \frac{r^2 - |w(z)|^2}{1-r^2}.$$

If we take

$$p(z) = \frac{1 + dw(z)}{1 - bw(z)} \tag{2.10}$$

then

$$|p(z) - A| \leq B \tag{2.11}$$

where

$$A = \frac{1 + bdr^2}{1 - b^2r^2} \tag{2.12}$$

and

$$B = \frac{(b + d)r}{1 - b^2r^2}. \tag{2.14}$$

Substituting value of $w(z)$ from (2.10) in (2.9) we get

$$\Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq \frac{1}{b + d}\left[E - d\Re\left\{\frac{1}{p(z)}\right\} + (a - 2b)\Re\{p(z)\} - \right. \\ \left. - \frac{r^2|bp(z) + d|^2 - |p(z) - 1|^2}{(1 - r^2)|p(z)|}\right]. \tag{2.14}$$

If we take $p(z) = A + u + iv$, $|p(z)| = R$ and use (2.12) and (2.13) in (2.14) we get

$$\Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} \geq \frac{1}{b + d}\left[E - \frac{d(A + u)}{R^2} + (a + 2b)(A + u) - \right. \\ \left. - \frac{B^2 - u^2 - v^2}{R}\left(\frac{1 - b^2r^2}{1 - r^2}\right)\right] \equiv \frac{1}{b + d} \cdot P(u, v). \tag{2.15}$$

Differentiating $P(u, v)$ partially with respect to $v$ we get

$$\frac{\delta P(u, v)}{\delta v} = \frac{v}{R}\left[\frac{2d(A + u)}{R^3} + \left\{2 + \frac{B^2 - u^2 - v^2}{R}\right\}\left(\frac{1 - b^2r^2}{1 - r^2}\right)\right]. \tag{2.16}$$

If $d \geq 0$, quantity in the square brackets is positive. If $d < 0$ we see that

$$\frac{1 - b^2r^2}{1 - r^2} + \frac{d(A + u)}{R^3} \geq 1 + \frac{d(1 + br)^2}{(1 - dr)^2} \geq 0$$

and therefore the quantity in the square brackets in (2.16) is positive.

So $\dfrac{\delta P(u,v)}{\delta v} \geq 0$ if $v \geq 0$ and $\dfrac{\delta P(u,v)}{\delta v} < 0$ if $v < 0$ therefore

$$\min_v P(u,v) = P(u,0) = \tag{2.17}$$

$$= E - \frac{d}{R} + (a+2b)R - \frac{B^2-(R-A)^2}{R}\left(\frac{1-b^2r^2}{1-r^2}\right) \equiv P(R)$$

where $R = A + u$.

$P'(R)$ is an increasing function of $R$ and $P'(R_0) = 0$ where

$$R_0 = \left[\frac{(1-d)(1+dr^2)}{(a+2b+1)-(a+2b+b^2)r^2}\right]^{1/2}. \tag{2.18}$$

Again we see that $P'(A+B) \geq 0$ therefore $R_0 \leq A+B$. Since $P'(R)$ is increasing function of $R$ and $A-B \leq R \leq A+B$ we have

$$\min_R P(R) = \begin{cases} P(A-B) & \text{if } 0 \leq R_0 \leq A-B \\ P(R_0) & \text{if } A-B \leq R_0 \leq A+B. \end{cases} \tag{2.19}$$

$$= \begin{cases} \dfrac{(b+d)[(1-\beta)-\{\beta(b-d)+a+b+2d\}r+d(a+b\beta)r^2]}{(1-dr)(1+br)} & \text{if } R_0 \leq A-B \\ (E-1+bd)-(1+bd)x+ & \\ +\sqrt{(1-d)\{(1-d)+(1+d)x\}\{(1+2a+4b+b^2)+(1-b^2)x\}} & \text{if } R_0 \geq A-B \end{cases}$$

where $x = \dfrac{1+r^2}{1-r^2}$.

Let us take

$$Q(r) = (A-B)^2 - R_0^2 = \left(\frac{1-dr}{1+br}\right)^2 - \frac{(1-d)(1+dr^2)}{(a+2b+1)-(a+2b+b^2)r^2}. \tag{2.20}$$

Therefore $Q(r)$ is a decreasing function of $r$ and

$$Q(0) = \frac{(a+b)+(b+d)}{(a+b)+(1+b)} \geq 0 \quad \text{and} \quad Q(1) = -\frac{2(1-d)(b+d)}{(1+b)(1-b^2)} \leq 0.$$

Therefore $Q(r)$ has unique root in (0, 1).

Let it be $r(a,b)$. Hence if $r \leq r(a,b)$, $Q(r) \geq 0$ i.e. $A-B \geq R_0$ and if $r \geq r(a,b)$, $Q(r) \leq 0$ i.e. $A-B \leq R_0$. So from (2.19) and (2.20) the result follows.

The equality in (2.4) is attained for the function $F(z) = z(1-bz)^{-\frac{a+b}{b}}$ and that in (2.5) for the function

$$F(z) = z(1-2kbz+b^2z^2)^{-\frac{a+b}{2b}}$$

where $k$ is given by

$$\frac{1+k(a-b)r-br^2}{1-2kbr+b^2r^2}=\left\{\frac{(1-d)(1+dr^2)}{(a+2b+1)-(a+2b+b^2)r^2}\right\}^{1/2}.$$

Similarly by using the method of theorem 1 following theorems follow.

THEOREM 2. *If $f(z)$ is regular in $D$ and satisfy*

$$F(z)=\left(\frac{c+2}{z^{c+1}}\right)\int_0^z t^{c-1}f(t)g(t)dt, \qquad c\geq 0$$

*where $F\in S^*(\beta)$ and $g\in S(m,M)$ then $f(z)$ is univalent and starlike of order $\beta$ in $|z|<r_0$ where $r_0$ is the smallest positive root of the equation*

$$\begin{aligned}(1-\beta)(c+2)-\{(c+2)(a+2b-b\beta)+2(1-\beta)(2-\beta)\}r+\{2b(1-\beta)(2-\beta)-\\ -(1-\beta)(c+2\beta)-2(c+1+\beta)(a+b)\}r^2-(c+2\beta)(a+b\beta)r^3=0.\end{aligned}$$

*This result is sharp.*

THEOREM 3. *If $f(z)$ is regular in $D$ and satisfies*

$$F(z)=\left(\frac{c+2}{z^{c+1}}\right)\int_0^z t^{c-1}f(t)g(t)dt, \quad c\geq 0$$

*where $F\in S^*(\beta)$ and $g\in K(\alpha)$, then $f(z)$ is starlike of order $\beta$ for $|z|<r_0$, where $r_0$ is the smallest positive root of the equation*

$$\begin{gathered}(c+2)(2-\beta)+2\{(c+\beta+1)-(1-\beta)(2-\beta)\}r+\beta(c+2\beta)r^2-(1+r)\\ \{(c+2)+(c+2\beta)r\}B(\alpha,r)=0\end{gathered}$$

*where*

$$B(\alpha,r)=\begin{cases}\dfrac{(2\alpha-1)r}{(1-r)^{2(1-\alpha)}\{1-(1-r)^{2\alpha-1}\}}, & \alpha\neq\dfrac{1}{2}\\[2ex] -\dfrac{r}{(1-r)\log(1-r)}, & \alpha=\dfrac{1}{2}\end{cases}$$

*This result is sharp.*

THEOREM 4. *If $f(z)$ is regular in $D$ and satisfy*

$$F(z)=\left(\frac{c+2}{z^{c+1}}\right)\int_0^z t^{c-1}f(t)g(t)dt, \quad c\geq 0$$

*where $F \in S^*(\beta)$ and $g(z)/z \in P(\alpha)$ then $f(z)$ is univalent and starlike of order $\beta$ in $|z| < r_0$ where $r_0$ is the smallest positive root of the equation*

$$(c+2)(1-\beta) - 2\{(c+2)(1-\alpha\beta) + (1-\beta)(2-\beta)\}r - 2\{c(3-4\alpha-\beta+\alpha\beta)+ \\ +(3+2\beta-8\alpha+6\alpha\beta-\beta^2-2\alpha\beta^2)\}r^2 + 2\{(c+2\beta)(2\alpha-\alpha\beta-1)-(2\alpha-1) \\ (1-\beta)(2-\beta)\}r^3 - (2\alpha-1)(1-\beta)(c+2\beta)r^4 = 0.$$

*The result is sharp.*

Theorem 5. *Let $F \in \Gamma(m, M)$ ande $f(z)$ be defined by*

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt, \quad c \geq 1$$

*and $r(a,b)$ be the unique positive root of the equation*

$$(a+d) + 2\{d(a+b)-(d-b)\}r + \{2(b^2-d^2)-(a+d)+d(ad+b^2)\}r^2 - \\ -2d\{(a+b)+b(d-b)\}r^3 - d(ad+b^2)r^4 = 0$$

*and $d \leq 0$ then $f(z)$ is meromorphic starlike of order $\beta$ for $|z| < r_0$, where $r_0$ is the smallest positive root of the equation*

$$(1-\beta) + \{(a+b+2d)-(b+d)\beta\}r + (ab+bd+d^2-bd\beta)r^2 = 0$$

*if $0 < r_0 \leq r(a,b)$, and that of the equation*

$$(E-1+bd)-(1+bd)x+\sqrt{(1+d)\{(1+d)+(1-d)x\}\{(1-2a+b^2)+(1-b^2)x\}}$$

*if $r(a,b) \leq r_0$ where*

$$x = \frac{1+r^2}{1-r^2}, \quad E = (a-b)-(d-b)\beta \;\; \text{and} \;\; d = \frac{a+b+c}{c}.$$

*Equality is attained for the functions*

$$F(z) = \frac{(1+bz)^{\frac{a+b}{b}}}{z}$$

$$F(z) = \frac{[(1-bz)^{1+k}(1+bz)^{1-k}]^{\frac{a+b}{2b}}}{z}$$

*where $k$ is determined from*

$$\frac{1-k(a+b)z+abz^2}{1-b^2z^2} = \left\{\frac{(1+d)(1-dr^2)}{(1-a)+(a-b^2)r^2}\right\}^{1/2}.$$

## REFERENCES

[1] S.K. Bajpai and R.S.L. Srivastava, *On the radius of convexity and starlikeness of univalent functions*, Proc. Amer. Math. Soc. **32** (1972), 153–160.

[2] P.L. Bajpai and Prem Singh, *The radius of starlikeness of certain analytic functious in the unit disc*. Proc. Amer. Math. Soc. 44 (1974), 395–402.

[3] Z.J. Jakubowski, *On the coefficients of starlike functions of some classes*, Annales Polon Matö. **26** (1972), 305–313.

Departamento de Matemática
I. C. E. – Universidade de Brasilia
Brasilia – D. F. – 70.910
BRASIL
I. I. T. Kanpur
Dept. Mat.