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FIXED POINTS OF MONOTONE MULTIFUNCTIONS IN PARTIALLY ORDERED SETS

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1. Introduction

Since Tarski's clasic result for increasing functions on lattices (see [12]), a number of papers giving fixed point theorems for increasing or decreasing functions on partially ordered sets have appeared in the last two decades. Some of them treat only increasing functions, for example, [4], [14], some only decreasing, for example, [1], [2], [3], [5], [8], and some increasing and decreasing, for example [13]. In [6] are considered so called "partly decreasing functions". Several papers treat fixed points of multifunctions ([7], [9] — [11]).

In the following $(P, \leqslant)$ is a non-empty set with a partial order $\leqslant$. A subset C of P is called a chain in case it is totally ordered. A multifunction $F: X \to Y$ is a point to set correspondence on X into Y with $F(x) \neq \varnothing$ for all $x \in X$, where $\varnothing$ denotes the empty set. We shall denote a multifunction by upper case latter, F, G ect. and a single — valued function (or simply function) by a lower case latter.

If $f: P \to P$ is a function, a *fixed point* of f is a point $x \in P$ such that $f(x) = x$, and if F is a multifunction, then x is a fixed point if $x \in F(x)$.

2. Decreasing multifunctions

Recall that a single — valued function $F: P \to P'$, P and P' are partially ordered sets, is decreasing if $x_1 \leqslant x_2$, $x_1, x_2 \in P$, implies $f(x_1) \geqslant f(x_2)$. There are many ways to generalize the notion of decreasing function to a notion of decreasing multifuction. (Add that a decreasing function is also called antitone function.) For example, in [7], the following definition is given.

Let P be a partially ordered set. Then a multifunction $F: P \to P$ is decreasing if $x \leqslant y$ implies

1. For all $z \in F(x)$, $L(z) \cap F(y) \neq \varnothing$.

2. For all $z \in F(y)$, $M(z) \cap F(x) \neq \varnothing$.

$(L(z) = \{u \mid u \leqslant z\},\ M(z) = \{u \mid u \geqslant z\}.)$

The following two kinds of decreasing functions in complete lattices and conditionally complete partially ordered sets are considered in [2], [3], [5], [8] and [15].

Let L be a complete lattice and $f: L \to L$ a function. If, for any $\varnothing \neq A \subset L$,

$$f(\sup A) = \inf f(A), \text{ where } f(A) = \{f(a) \mid a \in A\},$$

then we say that f is a *join antimorphism*.

A *meet antimorphism* is defined in a similar way, namely, for $\varnothing \neq A \subset L$,

$$f(\inf A) = \sup f(A).$$

It is easily seen that join antimorphisms and meet antimorphisms are decreasing functions.

In a complete lattice we introduce here the following types of decreasing multifunctions for complete lattices.

Inf — decreasing multifunctions. Let L be a complete lattice and $F: L \rightsquigarrow L$ a multifunction. We say that F is inf-decreasing if (and only if), for any $a, b \in L$,

$$a \leqslant b \Rightarrow \inf F(b) \leqslant \inf F(a) \tag{1}$$

Let us consider the following two conditions:

$$\text{for every } x \in L, \ \inf(F(\inf F(x)) \geqslant x; \tag{2}$$

$$\text{for every } x \in L, \ \inf F(x) \in F(x). \tag{3}$$

Definition 2.1. Let L be a complete lattice and let $F: L \to L$ be a multifunction. We say that F is a *join d-multimorphism* if (and only if), for any $A \subset L$ $(A \neq \varnothing)$

$$\inf F(A) \in F(\sup A), \tag{4}$$

where $F(A) = \cup\{F(a) \mid a \in A\}$.

A multifunction $F: L \to L$ is a *meet d-multimorpism* if $\varnothing \neq A \subset L$ implies

$$\sup F(A) \in F(\inf A). \tag{4'}$$

Proposition 2.1. *Let L be a complete lattice and let $F: L \to L$ be a join d-multimorpism, then F is inf-decreasing on L.*

Proof. Suppose $a, b \in L$ and $a \leqslant b$. Let $m = \inf(F(a) \cup F(b))$. Then $F(\sup\{a, b\}) = F(b)$ and

1° $n = \inf F(b) \in F(b)$ (by (4) for $A = \{b\}$)

2° $m \in F(b)$.

Since $F(b) \subset F(a) \cup F(b)$, it follows

$$m \leqslant n \tag{*}$$

But, by 2°,

$$n \leqslant m \tag{**}$$

(*) and (**) imply $m = n$. Hence inf $F(a) \geqslant m$, or inf $F(a) \geqslant \inf F(b)$.

We see that every join d-multimorphism satisfies (1) and (3).

Since every constant single — valued mapping satisfies (4), we conclude that the condition (2) is not implied by (4) (see [2]). On the other hand, the following proposition is valid.

Proposition 2.2. *Let L bn a complete lattice and let $F:L\to L$ be a multifunction satisfying the conditiovs* (1) — (3). *Then F is a join d-multimorphism.*

Proof. Let $\varnothing\neq A\subset L$ and $s=\sup A$. Then, by (1), $\inf F(x)\geqslant\inf F(s)$ for every $x\in A$, that is $\inf F(s)$ is a lower bound for $F(A)$. Let $m=\inf F(A)$. Evidently

$$(\circ)\qquad m\geqslant\inf F(s)$$

From $\inf F(x)\geqslant m$, it follows, by (1), that $\inf F(\inf F(x))\leqslant\inf F(m)$, or, by (2), $x\leqslant\inf F(m)$. We conclude that $\inf F(m)$ is an upper bound for A. Then $\inf F(m)\geqslant s$, which implies $\inf F(\inf F(m))\leqslant\inf F(s)$, or, again by (2),

$$(\square)\qquad m\leqslant\inf F(s).$$

($\circ$) and ($\square$) imply

$$(\mathrm{I})\qquad \inf F(A)=\inf F(\sup A)$$

But by (3), $\inf F(s)\in F(s)$, which proves the assertion.

The following example will show that a multifunction $F:L\to L$ may satisfy (1) and (2) without satisfying (4), or, without being a join d-multimorphism.

Example 2.1. Let L be the lattice on the Figure 1 and let $F:L\to L$ be defined by: $F(0)=\{1\}$, $F(a)=F(b)=\{0\}$, $F(1)=\{a, b, 1\}$. It is easy to verify that F satisfies (1) and (2), but not (3).

On the other hand one can easily construct a single — valued mapping of L into itself, which satisfies (1) and (3) but not (2) and which is not a single — valued join antimorphism.

Also, the identity mapping of L satisfies (2) and (3) but not (1).

Let us note that we shall write $a\vee b$ (resp. $a\wedge b$) instead of $\sup\{a, b\}$ (resp. $\inf\{a, b\}$).

By an *antichain* of partically ordered set P we mean a subset A of P such that no two elements of A are combarable.

Our first fixed point theorem concerns complete lattices satisfying the following condition.

Condition (α). *There is an antichain $A\subset L$ (where L is a complete lattice) such that every $x\in L$ is either a join or a meet of elements of A.*

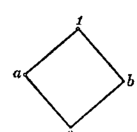

Figure 1

Theorem 2.3. *Let L be a complete lattice satisfying the condition (α) (with the antichain A) and let $F:L\to L$ be an inf-increasing multifunction such that:*

(i) *Inf* $F(x)\in F(x)$, *for every* $x\in L$;

(ii) $x\leqslant$ *inf* $F($*inf* $F(x))$, *for every* $x\in L$;

(iii) *for every* $a\in A$, *inf* $F(a)$ *is comparabte to* a;

(iv) $\{a\in A \mid a\leqslant \mathit{inf}\ F(a)\}\neq\varnothing$

Then F has a fixed point.

Proof. Let us denote by $P_F(L)$ the family of all subsets B of L such that sup $B \leqslant \inf F(\sup B)$. By the condition (iv), $P_F(L)$ is noe-empty.

Lemma 2.4. Let L be a complete lattice aad let $F: L \to L$ be a join d-multimorpism, then $P_F(L)$ has a maximal element.

Proof of the lemma. Let $(B_i \mid i \in I)$ be a chain $P_F(L)$. We shall show that $B_0 = \bigcup \{B_i \mid i \in I\}$ belongs to $P_F(L)$. Given any $i, j \in I$. We may assume $B_i \subset B_j$, hence sup $B_i \leqslant$ sup B_j. Put sup $B_i = b_i$, sup $B_j = b_j$, inf $F(b_i) = m_i$, inf $F(b_j) = m_j$. We have

$$b_i \leqslant b_j \leqslant m_j \leqslant m_i \text{ (since } \sup B_j \leqslant \inf F(\sup B_j) \text{ and so for } B_i)$$

It follows that sup $\{b_i \mid i \in I\} \leqslant m_j$, for every $j \in I$ and so

$$\sup \{b_i \mid i \in I\} \leqslant \inf \{m_j \mid j \in I\} \tag{1.1}$$

Remark that $\sup \{b_j \mid j \in I\} = \sup B_0$. By (1.1) we have

$\sup B_0 = \sup \{b_i \mid i \in I\} \leqslant \inf \{m_j \mid j \in I\} = \inf F(B')$, where $B' = \{b_i \mid i \in I\}$.

Let us prove the last equality.

Put $m = \inf \{m_j \mid j \in I\}$, $n = \inf F(\{b_j \mid j \in I\})$.

$$b_j \in B' \text{ implies } F(b_j) \subset F(B'),$$

and so $\inf F(b_j) \geqslant \inf F(B')$, or $m_j \geqslant n$ for $j \in I$, hence

$$m \geqslant n \tag{$'$}$$

Let $x \in F(B')$. Then $x \in F(b_j)$ for some $j \in I$. Hence $x \geqslant m \geqslant m$ or

$$n \geqslant m \tag{$''$}$$

($'$) and ($''$) prove the desired equality.

But sup $B' = b_0$, hence $\sup B_0 \leqslant \inf F(B') = \inf F(b_0) = \inf F(\sup B_0)$, since F satisfies the relation (I) from the proof of the Proposition 2.2, and $b_0 = \sup B_0$.

The assertion of the lemma follows now from Zorn's lemma.

Proof of the theorem. By the proposition 2.2, F is a join d-multimorphism, so we may apply the lemma 2.4. Let B_0 be a maximal element of $P_F(L)$ and $b_0 = \sup B_0$. Then $\inf F(b_0) \geqslant b_0$. Suppose $\inf F(b_0) > b_0$. Since L satisfies (α), there exists $a \in A$, noncomparable to b_0, such that $\inf F(b_0) > a$.

Case 1°

$$\inf F(a) \geqslant a \tag{1.2}$$

From $\inf F(b_0) > a$ it follows that $\inf F(\inf F(b_0)) \leqslant \inf F(a)$ since F is inf-decreasing). By the condition (ii) of the theorem,

$$b_0 \leqslant \inf F(a) \tag{1.3}$$

From (1.2) and (1.3) we obtain $\inf F(a) \geqslant b_0 \vee a$. Since $\inf F(b_0) \geqslant b_0 \vee a$ and $\inf F(a) \geqslant b_0 \vee a$, we have

$$\inf F(b_0) \wedge \inf F(a) \geqslant b_0 \vee a.$$

But $\inf F(b_0) \wedge \inf F(a) = \inf(F(b_0) \cup F(a))$. Hence $\inf F(\{a, b_0\}) \geqslant a \vee b_0$ and so $\inf F(B_0 \cup \{a\}) \geqslant \inf F(b_0 \vee a) \geqslant b_0 \vee a$, contradicting maximality of B_0.

Case 2°

$$\inf F(a) < a$$

Then from $\inf F(b_0) \geqslant b_0 \vee a$, since F is inf-decreasing and because of (ii), it follows

$$b_0 \leqslant \inf F(\inf F(b_0)) \leqslant \inf F(b_0 \vee a).$$

But by (I)

$$\inf \{F(b_0) \cup F(a)\} = \inf F(b_0 \vee a),$$

and so

$$b_0 \leqslant \inf \{F(b_0) \cup F(a)\} = \inf F(b_0) \wedge \inf F(a) < a,$$

contradicting non-comparability of b_0 and a.

So in both cases $\inf F(b_0) = b_0$. But $\inf F(b_0) \in F(b_0)$, i.e. $b_0 \in F(b_0)$.

Theorem is proved.

Corollary 2.5. *Let L be a complete lattice satisfying the condition* (α) *and let $f: L \to L$ be a decreasing function such that:*

(i) $f^2(x) \geqslant x$, *for every* $x \in L$ (*where* $f^2 = f \circ f$);

(ii) *for every* $a \in A$, $f(a)$ *is comparable to* a;

(iii) $\{a \in A \mid a \leqslant f(a)\} \neq \varnothing$..

Then f has a fixed point.

A complete lattice L is said to be atomic if every $x \in L$, $x \neq 0$, is a join of atoms (elements of L that cover 0). Denote by A the set of atoms of an atomic lattice L. Then we have the following corollary.

Corollary 2.6. *Let L be a complete atomic lattice and let $f: L \to L$ be a decreasing function such that:*

(i) *For every* $x \in L$, $f^2(x) \geqslant x$;

(ii) $f(a)$ *is comparable to* a *for every* $a \in A$.

Then f has a fixed point.

Corollary 2.5. is improved version of the following theorem due to Shmuely.

Corollary 2.7. ([8], *theorem* 1) *Let L be a complete atomic lattice and let $f: L \to L$ be a decreasing function such that $f^2(x) \geqslant x$, for every $x \in L$. If $f(a) \geqslant a$, for every atom a, then f has a fixed point.*

Simple examples, similar to the examples [2] show that no one of the conditions of theorem 2.3 can be omitted.

Sup-decreasing multifunctions. Let L be a complete lattice and let $F: L \to L$ be a multifunction. We say that F is sup-decreasing if, for every $a, b \in L$,

(1′) $$a \leqslant b \Rightarrow \sup F(b) \leqslant \sup F(a)$$

Let us consider the following two conditions:

(2′) For every $x \in L$, $\sup F(\sup F(x)) \leqslant x$

(3′) For every $x \in L$, $\sup F(x) \in F(x)$

The following two propositions are proved analogously as propositions 2.1 and 2.2 and their proofs will be omitted.

Proposition 2.8. *Let L be a complete lattice and let $F: L \to L$ be a meet d-multimorphism. Then F is sup-decreasing.*

Proposition 2.9. *Let L be a complete lattice and let $F: L \to L$ be a multifunction satisfying the conditions (1′) — (3′). Then F is a meet d-multimorphism.*

An example analogous to the example 2.1 shows that a multlfunction $F: L \to L$ (where L is a complete lattice) may satisfy (1′) and (2′) without being a meet d-multimorphism.

Theorem 2.10. *Let L be a complete lattice satisfying the Condition (α) and let $F: L \to L$ be a sup-decreasing multifunction such that:*

(i) *For every $x \in L$, $\sup F(x) \in F(x)$;*

(ii) *For every $x \in L$, $\sup F(\sup F(x)) \leqslant x$;*

(iii) *For every $a \in A$, $\sup F(a)$ is comparable to a;*

(iv) $\{a \in A \mid a \geqslant \sup F(a)\} \neq \varnothing$

Then F has a fixed point.

Proof of this theorem is analogous to the proof of theorem 2.3 and will be omitted.

Corollary 2.11. *Let L be a complete lattice satisfying (α) and let $f: L \to L$ be a decreasing function such that:*

(i) *For every $x \in L$, $f^2(x) \leqslant x$;*

(ii) *For every $a \in A$, $f(a)$ is comparacle to a;*

(iii) $\{a \in A \mid a \geqslant f(a)\} \neq \varnothing$.

Then f has a fixed point.

A complete lattice L is coatomic provided that every $x \in L$ ($x \neq 1$) is a meet of coatoms (elements of L covered by 1). Denote by C the set of coatoms of L.

Corollary 2.12. *Let L be a complete coatomic lattice and let $f: L \to L$ be a decreasing function such that:*

(i) *$x \geqslant f^2(x)$, for every $x \in L$;*

(ii) *$f(c)$ is comparable to c for every $c \in C$.*

Then f has a fixed point.

3. Some increasing multifunctions

We recall that a single — valued function $f: P \to P$ (where P is a partially ordered set) is increasing if $x_1 \leqslant x_2$, $x_1, x_2 \in P$, then $f(x_1) \leqslant f(x_2)$. There are many ways to generalize the notion of increasingness from single — valued functions to multifunctions. For example, R. E. Smithson (see [9] — [11]) stated the following.

Condition I. If $x_1 \leqslant x_2$, $x_1, x_2 \in P$, and $y_1 \in F(x_1)$, then there is a $y_2 \in F(x_2)$ such that $y_1 \leqslant y_2$.

We give the following generalization of the notion of increasing function on partially ordered sets.

Definition 3.1. *We say that a multifunction $F: P \to P'$, where P and P' are non-empty posets, is inf-increasing (resp. sup-increasing) if, for any $x \in P$, inf $F(x)$ (resp. sup $F(x)$) exists and $x \leqslant y$ implies inf $F(x) \leqslant$ inf $F(y)$ (resp. sup $F(x) \leqslant$ sup $F(y)$).*

It is evident that a single — valued function, considered as a multifunction, satisfies the condition I as well as the conditions of the definition 3.1.

We shall first establish a connection between Condition I and definition 3.1.

Proposition 3.1. *Let P be a non-empty poset and let $F: P \to P$ be a multifunction. Then the following two conditions are equivalent:*

(a) *F satisfies the Condition I and, for any $x \in P$,* sup $F(x) \in F(x)$;

(b) *F is sup-increasing and* sup $F(x) \in F(x)$, *for every $x \in P$.*

Proof. Let F satisfy (a) and let $x_1 \leqslant x_2$, $x_1, x_2 \in P$. Then, since the Condition I is fulfilled, for $y_1 = \sup F(x_1)$, there exists a $y_2 \in F(x_2)$ such that $y_1 \leqslant y_2$. But $y_2 \leqslant \sup F(x_2)$, so $\sup F(x_1) \leqslant \sup F(x_2)$, i.e. F is sup-increasing.

Conversely, let F be sup-increasing, $x_1 \leqslant x_2$, $x_1, x_2 \in P$, and $y_1 \in F(x_1)$. Then $y_1 \leqslant \sup F(x_1) \leqslant \sup F(x_2)$ and the role of y_2 in the Condition I has $\sup F(x_2)$. So the Condition I is satisfied and also $\sup F(x) \in F(x)$ for any $x \in P$.

The proposition 3.1. has , at once, as a corollary a theorem of Smithson. To state that theorem, we need the definition of a selection. Let $F: X \to Y$ be a multifunction on X into Y. A *selection* for F is a (single — valued) function $f: X \to Y$, such that $f(x) \in F(x)$ for each $x \in X$. An isotone (increcsing) is a selection which is isotone.

Corollary 3.2. ([9], *theorem* 1.7). *Let P be a poset and let $F: P \to P$ be a multifunction on P which satisfies Condition I. If* sup $F(x) \in F(x)$ *for all $x \in P$, then there is an isotone selection for F.*

In [9] R. E. Smithson gave another generalisation of incresing single — valued function.

Condition II. If $x_1, x_2 \in P$, $x_1 \leqslant x_2$ and if $y_2 \in F(x_2)$, then there is a $y_1 \in F(x_1)$ such that $y_1 \leqslant y_2$.

The following proposition is valid (we omit the proof).

Proposition 3.3. *Let P be a non-void poset and let $F:P\to P$ be a multifunction. Then the following two conditions are equivalent*:

(a) *F satisfies Condition II and fo· any $x\in P$,* $\inf F(x)\in F(x)$;

(b) *F is inf-increasing and* $\inf F(x)\in F(x)$ *for each $x\in P$.*

Now we give a multifunction version of Tarski's theorem.

Theorem 3.4. *Let L be a complete lattice and let $F:L\to L$ be an inf-increasing multifunction, such that, for any $x\in L$,* $\inf F(x)\in F(x)$. *Then F has a fixed point.*

Proof. Let $L^F=\{x\in L \mid x\leqslant \inf F(x)\}$ and $s=\sup L^F$. By well known procedure, by which Tarski's theorem is proved, we find that $s=\inf F(s)$. But $\inf F(s)\in F(s)$, hence $s\in F(s)$.

Remark 3.1. An analogous theorem is valid for sup-increasing multifunctions.

Remark 3.2. If F is a single — valued function, we have the theorem of Tarski so the theorem 3.4 is a generalisation of Tarki's theorem.

Remark 3.3. The following example will show that the condition $\inf F(x)\in F(x)$ cannot be omitted in the theorem 3.4. Let L be the lattice on the Figure 1, and let $F:L\to L$ be defined by:

$$F(0)=F(b)=F(1)=\{a\},\ F(a)=\{b,1\}.$$

Evidently F is inf-increasing, L is a complete latt,ce, but no fixed point of F exists.

4. Commuting famiiies of increasing multifunctions

Lep P be a partially ordered set and let $\mathcal{F}$ be a family of multifunctions on P into itself. We say that $\mathcal{F}$ is a commuting family if, for any $F, G\in\mathcal{F}$, $F\circ G=G\circ F$, where $(F\circ G)(x)=\bigcup\{F(u)\mid u\in G(x)\}$. A family $\mathcal{F}$ is inf-commuting if, for any $x\in P$ and any $F, G\in\mathcal{F}$, $\inf F(\inf G(x))=\inf G(\inf F(x))$, provided the corresponding inf ma always exist. (In this section we assume that $\inf F(x)\in F(x)$ for any $x\in P$ and all F are inf-commuting and also inf--increasing mappings.)

Proposition 4.1. *Let F and G be two inf-increasing multifunctions on P into itself. If F and G are commuting and inf $(F\circ G)(x)$ exists for any $x\in P$, then they are inf-commuting.*

Proof. Put $m=\inf F(\inf G(x))$, $n=\inf G(\inf F(x))$. For any $u\in G(x)$ we have $\inf G(x)\leqslant u$. It follows, since F is inf-increasing, that $\inf F(\inf G(x))\leqslant$ $\leqslant \inf F(u)$, or $m\leqslant \inf F(u)$. This inequality is valid for all $u\in G(x)$, hence $m\leqslant \inf\{\inf F(u)\mid u\in G(x)\}$. But $\inf\{\inf F(u)\mid u\in G(x)\}=\inf\bigcup\{F(u)\mid u\in G(x)\}=$ $=\inf(F\circ G)(x)$, hence $m\leqslant \inf(F\circ G)(x)$. Since F and G contain their inf ma, $\inf F(\inf G(x))\in(F\circ G)(x)$, or $m\geqslant \inf(F\circ G)(x)$. It follows: $m=\inf(F\circ G)(x)$. In a similar way it is proved that $n=\inf(G\circ F)(x)$. Since $(F\circ G)(x)=(G\circ F)(x)$, we have $m=n$. q.e.d.

Theorem 4.2. *Let P be a non-empty partially ordered set and let $\mathcal{F}$ be a non-empty inf-commuting family of inf-increasing multifunctions on P into itself. If there exists an element c of P such that $c \leqslant \inf F(c)$, for all $F \in \mathcal{F}$ and if each chain in P containing c has a supremum in P, then there exists a in P such that $a \in F(a)$, for all $F \in \mathcal{F}$.*

Proof. Let $\mathcal{S}$ be the set of all chains in P which contain c and satisfy: If $x \in C \in \mathcal{S}$, then $x \leqslant \inf F(x)$, for all $F \in \mathcal{F}$. By Zorn's lemma there exists a maximal chain L in $\mathcal{S}$. Since $c \in L$, L is non-empty and $a = \sup L$ exists in P. We shall show that $a \in L$. Let $F \in \mathcal{F}$ and $x \in L$. Then $x \leqslant a$ and therefore $x \leqslant \inf F(x) \leqslant \inf F(a)$. Hence, $\inf F(a)$ is an upper bound for L and thus $a \leqslant \inf F(a)$. Suppose that $a < \inf F(a)$ for some $F \in \mathcal{F}$ and let $G \in \mathcal{F}$. Then $a \leqslant \inf G(a)$ and $\inf F(a) \leqslant \inf F(\inf G(a)) = \inf G(\inf F(a))$. It follows that $L \cup \{\inf F(a)\} \in \mathcal{S}$ which contradicts the maximality of L. Hence $\inf F(a) = a$ for all $F \in \mathcal{F}$. But $\inf F(a) \in F(a)$, and the theorem is proved.

Applying proposition 4.1 and theorem 4.2 we obtain the following

Theorem 4.3. *Let P be a partially ordered set and let $\mathcal{F}$ be a non--empty commuting family of inf-increasing multifunctions on P into itself, such that, for any $x \in P$, and any $F, G \in \mathcal{F}$, $\inf(F \circ G)(x)$ exists. If there exists an element c of P such that $c \leqslant \inf F(c)$ for all F and if each chain in P containing c has a supremum in P, then there exists $a \in P$ such that $a \in F(a)$ for all $F \in \mathcal{F}$.*

As a corollary we obtain an extension of Tarski's theorem ([12]).

Corollary 4.4. *Let P be a complete lattice and let $\mathcal{F}$ be commuting family of inf-icreasing multifunctions such that $\inf F(x) \in F(x)$ for each $F \in \mathcal{F}$ and for all $x \in P$, then there exists a common fixed point for the members of $\mathcal{F}$.*

In an analogous way the following theorem is proved.

Theorem 4.5. *Let P be a partially ordered set and let $\mathcal{F}$ be a non--empty commuting family of sup-increasing multifunctions on P into itself, such that, for all $x \in P$ and any $F, G \in \mathcal{F}$ $\sup(F \circ G)(x)$ exists. If there exists an element c of P such that $c \leqslant \sup F(c)$, for each $F \in \mathcal{F}$ and if each chain in P containing c has a supremum in P, then there exists $a \in P$ such that $a \in F(a)$ for all $F \in \mathcal{F}$.*

Applying proposition 3.1 and theorem 4.5, we obtain the following theorem of R. E. Smithson (see [11], theorem 2.3).

Corollary 4.6. *Let $c \in P$ and suppose each chain containing c has a supremum in P. Let $\mathcal{F}$ ce a commuting collection of multifunctions on P into itself, such that there exists $y \in F(c)$ with $c \leqslant y$ for each $F \in \mathcal{F}$. If each $F \in \mathcal{F}$ satisfies Condition I, and if $\sup F(x) \in F(x)$ for each $F \in \mathcal{F}$ and for all $x \in P$, then there exists $x \in P$ such that $x \in F(x)$ for all $F \in \mathcal{F}$.*

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