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CONJUGATELY FACTORABLE ISOTONE SELF-MAPPINGS OF COMPLETE LATTICES

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1. Let P be a partially ordered set. A function $f\colon P\to P$ will be called *isotone* if

(2) $$a\leqslant b \text{ implies } f(a)\leqslant f(b)$$

and *antitone* if

(2) $$a\leqslant b \text{ implies } f(a)\geqslant f(b)$$

for any $a, b\in P$.

In a complete lattice P one considers a function f such that for any $\varnothing\neq A\subset P$

(3) $$f(\vee A)=\wedge f(A), \text{ where } f(A)=\{f(a)\mid a\in A\}.$$

A function f in a comlete lattice P satisfying (3) is referred to as a *join antimorfhism.*

It is easily seen that every function f in a complete lattice P satisfying (3) aso satisfies (2), that is: any join antimorphism is an antitone mapping. On the other hand, it is easy to construct an antitone mapping of a comlete lattice P into itself which is not a join antimorphism.

Notations. Let P be a set and $f\colon P\to P$. Put

$$I(f(P)=\{x\mid x\in P \text{ and } f(x)=x\},$$

.e. $I(f, P)$ is the set of all fixed points of f.

Instead of $f\circ f$ we shall write f^2.

2. *Definitions and re.ults.* Definition. Let $g\colon P\to P$ be any isotone mapping. We say that g is factorable if there exist mappings $f_1, f_2\colon P\to P$ such that $g=f_1\circ f_2$. Two isotone mappings g and h are said to be conjugately factorable if there exist mappings $f_1, f_2\colon P\to P$ such shat $g=f_1\circ f_2$ and $h=f_2\circ f_1$.

Theorem *Let P be a complete lattice and $g, h: P \to P$ two conjugately factorable isotone mappings (say $g = f_1 \circ f_2$, $h = f_2 \circ f_1$).*

Then

(i) $f_1(I(h, P)) \subset I(g, P)$

(II) $f_2(I(g, P)) \subset I(h, P)$.

Proof. Since P is a complete lattice and g, h isotone mappings of P into itself, both sets $I(g, P)$ and $I(h, P)$ are nonempty (see [3]).

Let $x \in I(h, P)$. Then $f_1(x) = f_1(h(x)) = (f_1 \circ (f_2 \circ f_1))(x) = ((f_1 \circ f_2) \circ f_1)(x) = (g \circ f_1)(x) = g(f_1(x))$. Hence $f_1(x) \in I(g, P)$, which proves (i).

Similarly is shown the inclusion (ii).

Since, by Tarski's theorem, $I(g, P)$ (resp. $I(h, P)$) is a complete lattice, $m = \inf I(g, P)$ and $n = \inf I(h, P)$ exist, so from the above theorem we conclude.

Corollary 1. *(Blair and Roth, [1], theorem 4). Under the assumptions of the theorem there exist m and n such that*

$$m = g(m) \leqslant f_1(n), \quad n = h(n) \leqslant f_2(m).$$

For $f_1 = f_2 = f$, where f is a join antimorpism, we have

$$f(I(f^2, P)) \subset I(f^2, P)$$

and if we put $m = \inf I(f^2, P)$. then $m = f^2(m) \leqslant f(m)$.

So we obtain

Corollary 2. *(Roth, A. E., [2]) Let P be a complete lattice and $f: P \to P$ a join antimorphism. Then there exists $m \in P$ such that $m = f^2(m)$ and $m \leqslant f(m)$.*

Remaque. A. E. Roth applied this theorem in studying two person's game (see references of [1] and [2]).

By simple reasoning is seen tha preceding corrolary 2 remains valid if f, instead of to be a join antimorphism, is an arbitrary antitone mapping, so we have the following generalisation of Roth's theorem.

Corollary 3. *Let P be a complete latice and $f: P \to P$ antitone. Then there exists $s \in P$ such that $s = f^2(s)$ and $s \leqslant f(s)$.*

Putting $M = \sup I((g, P)$, $N = \sup I(h, P)$, we obtain the following.

Corollary 4. *Under the assumptions of the theorem, there exist $M, N \in P$ such that*

$$M = g(M) \geqslant f_1 \quad N = h(N) \geqslant f_2(M).$$

Analogously we deduce

Corollary 5. *There exists an element M of P (where P is a complete lattice) such that $M=f^2(M)$ and $M \geqslant f(M)$, provided that $f: P \to P$ is any antitone mapping (a join antimorphism included).*

REFERENCES

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[3] Tarski, Alfred, *A lattice theoretical Fixpoint Theorem and its Aplications,* Pacific Journal of Mathematics, vol. 5, № 2, 1955.