PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 27 (41), 1980, pp. 33—36.

LOCALLY EVENTUALLY CONTRACTIVE FIXED-POINT MAPPINGS

Ljubomir B. Ćirić

(Communicated September 4, 1978.)

Let (M, d) be a metric space and T a mapping of M into itself. A mapping T is said to be locally contractive on M iff for each u in M there exists $S(u, r(u))=\{v: d(u, v)<r(u)\}$ such that $d(Tx, Ty)<d(x, y)$ for all x, y in $S(u, r(u))$, $x \neq y$. If $r(u) \geqslant \varepsilon>0$ for each u in M, then T is called ε-contractive. M. Edelstein [3], E. Rakotch [5] and S. Naimpally [4] have constructed some examples which show that locally contractive mappings may be without fixed or periodic points. M. Edelstein [3] has proved some fixed-point or periodic-point theorems of ε-contractive mappings which have a property that a sequence $\{T^n x\}_{n=0}^{\infty}$ contains a convergent subsequence for some x in M. In [2] is introduced a class of mappings which satisfy the following condition: for every x, y in M there exists an integer $m=m(x, y)$ such that

$$x \neq y \text{ implies } d(T^n x, T^n x)<d(x, y) \text{ for all } n \geqslant m(x, y). \tag{1}$$

Such mappings are called eventually contractive operators, and if they are orbitally continuous and $\{T^n x\}_{n=0}^{\infty}$ contains a convergent subsequence, then they have fixed-point.

In this paper a class of mappings which satisfy a locally contractive condition of the type (1) is introduced and fixed-point and periodic-point theorems for such mappings are proved.

Definition. *A selfmapping T of a metric space M into itself we call a locally eventually contractive mapping iff for each u in M there exists a spherical nbd $S(u, r(u))$ of u such that for every $x, y \in S(u, r(u))$, $x \neq y$, there exists a positive integer $m(x, y)$ such that* (1) *holds, i.e.*

$$d(T^n x, T^n y)<d(x, y) \textit{ for all } n \geqslant m(x, y).$$

It is clear that a class of locally eventually contractive mappings includes eventually contractive mappings with $r(u)=\infty$, and locally contractive mappings with $m(x, y)=1$. The following example shows that the discovered class of mappings is extensive.

Example. For each positive integer n let

$$M_n=\{x=(x_1, x_2): x_1=2^{1-n}\cos t,\ x_2=2^{1-n}\sin t: 0<t\leqslant 2\pi\}$$

be a subset of the Euclidean plane. Put $M=\bigcup_{n=1}^{\infty} M_n$ and let (M, d) be a metric space with usuall metric. Define $T: M\to M$ by

$$T(t)=2t, \text{ if } 0<t<\frac{\pi}{2},$$

$$=t+\frac{\pi}{2}, \text{ if } \frac{\pi}{2}\leqslant t<\pi$$

$$=\frac{t}{2}+\pi, \text{ if } \pi\leqslant t\leqslant 2\pi.$$

Then T is continuous and has infinite many fixed-poits: $(1, 0)$, $(2^{-1}, 0), \ldots,$ $(2^{1-n}, 0), \ldots$ Therefore, T is not eventually contractive on M. Also, T is not locally contractive at every point x with $t_x\leqslant\pi$ and $t_x=2\pi$. But T satisfies (1), as for every $u\in M_n$ we may put $r(u)=2^{-n}$ and then for $x, y\in S(u, 2^{-n})$ choose corresponding $m(x, y)$, which is large only in the case when $(2^{1-n}, 0)$ is between x and y.

Now we shall indicate sufficient condition for the existence periodic points of locally eventually contractive mappings.

Theorem 1. *Let T be a locally eventually contractive selfmapping of a metric space M into itself. If for some x in M the sequence $\{T^n x\}_{n=0}^{\infty}$ contains a convergent subsequence and T is orbitally continuous, then T has a periodic point in M.*

Proof. Let x and u in M be such that $\lim_{i\to\infty} T^{n_i}x=u$. Suppose that $T^n x\neq T^s x$ whenever $n\neq s$, since otherwise the Theorem follows. Choose fixed positive integers s and k such that $T^s x$ and $T^{s+k}x$ are in $S(u, r(u))$ and $d(T^s x, T^{s+k}x)<\frac{1}{3}r(u)$. Then by (1) for every $n\geqslant m(T^s x, T^{s+k}x)+s$ one has

$$(2)\qquad d(T^n x, T^{n+k}x)=d(T^{n-s}T^s x,\ T^{n-s}T^{s+k}x)<d(T^s x, T^{s+k}x)<\frac{1}{3}r(u).$$

Hence

$$\lim_{i\to\infty} d(T^{n_i}x, T^{n_i+k}x)=d(u, T^k u)\leqslant\frac{1}{3}r(u),$$

as $\lim_{i\to\infty} T^{n_i+k}x=T^k u$ by orbitally continuity of T. Therefore,

$$T^k u\in S\left(u, \frac{1}{3}r(u)\right)\subset S(u, r(u)).$$

Now we shall show that $T^k u = u$. Assume that $T^k u \neq u$. Since $\lim_{i\to\infty} T^{n_i} x = u$, $d(u, T^s x) < r(u)$ and $T^k u \in S(u, r(u))$, by (1) we may choose a positive integer $p \geqslant m(u, T^k u)$ such that $d(T^s x, T^{s+p} x) < \frac{1}{3} r(u)$ and

$$(3) \qquad d(T^p u, T^p T^k u) = d(T^p u, T^{p+k} u) < d(u, T^k u).$$

Let now $n \geqslant m(T^s x, T^{s+p} x) + s + m(T^s x, T^{s+k} x)$ be any positive integer such that $d(u, T^n x) < \frac{1}{3} r(u)$. Then

$$d(u, T^{n+p} x) \leqslant d(u, T^n x) + d(T^{n-s} T^s x, T^{n-s} T^{s+p} x)$$
$$< \frac{1}{3} r(u) + d(T^s x, T^{s+p} x) < \frac{2}{3} r(u),$$

$$d(u, T^{n+p+k} x) \leqslant d(u, T^{n+p} x) + p(T^{n+p-s} T^s x, \quad T^{n+p-s} T^{s+k} x)$$
$$< \frac{2}{3} r(u) + d(T^s x, T^{s+k} x) < r(u).$$

Therefore, for sufficiently large n

$$(4) \qquad T^{n+p} x, \quad T^{n+p+k} x \in S(u, r(u)) \text{ whenever } d(u, T^n x) < \frac{1}{3} r(u).$$

Since $\lim_{i\to\infty} T^{n_i} x = u$, there exists a positive integer N such that $d(u, T^{n_i} x) < \frac{1}{3} r(u)$ whenever $i > N$. Then for any fixed $n_i \geqslant m(T^s x, T^{s+p} x) + s + m(T^s x, T^{s+k} x)$ with $i > N$ we have

$$d(T^n x, T^{n+k} x) = d(T^{n-n_i-p} T^{n_i+p} x, T^{n-n_i-p} T^{n_i+p+k} x)$$
$$< d(T^{n_i+p} x, T^{n_i+p+k} x)$$

for every $n \geqslant m(T^{n_i+p} x, T^{n_i+p+k} x)$. Hence

$$(5) \qquad d(u, T^k u) \leqslant d(T^{n_i+p} x, T^{n_i+p+k} x),$$

as $d(u, T^k u)$ is a claster point of a sequence $\{d(T^n x, T^{n_i+k} x)\}_{n=1}^{\infty}$. But (5) implies

$$d(u, T^k u) \leqslant \lim_{i\to\infty} d(T^{n_i+p} x, T^{n_i+p+k} x) = d(T^p x, T^{p+k} x),$$

which is a contradiction with (3).

The proof is complete.

Now we bring sufficent condition for the existence of a fixed point. We recall that a mapping T is said to be asimptotically regular at x in M iff $\lim_{n\to\infty} d(T^n x, T^n y) = 0$.

Theorem 2. *Let T be a locally eventually contractive mapping and let for some x in M the sequence $\{T^n x\}_{n=0}^{\infty}$ contains a convergent subsequence. If T is orbitally continuous and asimptotically regular at x, then T has a fixed point.*

Proof. Let u in M be such that $\lim_{i\to\infty} T^{n_i} x = u$. Since T is asimptotically regular at x, there exists a positive integer K such that $n > K$ implies $d(T^n x, T^{n+1} x) < \frac{1}{3} r(u)$. Hence

$$\lim_{i\to\infty} d(T^{n_i} x, T^{n_i+1} x) = d(u, Tu) \leqslant \frac{1}{3} r(u).$$

Therefore, in this case $T^k u \in S(u, r(u))$ for $k = 1$. Then, as in Theorem 1, it follows that $Tu = u$. The proof is complete.

Note that if a locally eventually contractive mapping is not orbitally continuous, then T may be without periodic points. The following example shows it.

Example 2. Let $M = [0, 1]$ be a compact subset of the Euclidean plane. Define $T: M \to M$ by $Tx = \frac{x}{2}$, if $x \neq 0$ and $T(0) = 1$. Then T satisfies (1) with $r(u) = \text{diam}(M)$, $m(x, y) = 1$ for $x \neq 0$, $y \neq 0$ and $m(0, x) = E\left(\log_2 \frac{4}{x}\right)$ ($E(a)$ – the greatest integer not exceeding a). But T has not a periodic or fixed point.

REFERENCES

[1] Lj. Ćirić, *On locally contractive mappings and fixed points* (to appear).

[2] Lj. Ćirić, *Fixed and periodic points for a class of contractive operators*, Publ. Inst. Math., 18 (32) (1975), 57—69. MR 55 # 4121.

[3] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc., 37 (1962), 74—79. MR 24 # 2936.

[4] S. A. Naimpally, *A note on contractive mappings*, Indag. Math., 26 (1964), 275—279. MR 29 # 590.

[5] E. Rakotch, *A note on α-locally contrative mappings*, Bull. Res. Council Israel, F 10 F, (1962), 188—191. MR 26 # 4319.

[6] E. Rakotch, *On ε-contractive mappings*, Bull. Res. Concil Israel, F 10 F (1961), 53—58. MR 32 # 4683.

[7] B. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226 (1977), 257—290. MR 55 # 6406.