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A NOTE ON FIXED POINT MAPPINGS WITH CONTRACTING ORBITAL DIAMETERS

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1. Let T be a mapping of a metric space M into itself. For $x \in M$, let

$$O(x) = \{x, Tx, T^2 x, \ldots, T^n x, \ldots\}$$

and $r(x) = \operatorname{diam}(O(x))$.

Definition. If T has the property that for each $x \in M$ with $r(x) < \infty$ there exists $T^k x \in O(x)$ such that

$$r(T^k x) \leqslant \varphi[r(x)], \tag{1}$$

where $\varphi: R^+ \to R^+$ is a real function which is upper semicontinuous from the right and satisfies the inequality

$$\varphi(t) < t \text{ for every } t > 0,$$

then T will be said to have *contracting orbital diameters* on M.

In this note we shall investigate such mappings and prove a fixed-point theorem.

2. We shall prove the following result.

Theorem. *Let M be a complete metric space, T an orbitally continuous self-mapping on M which has contracting orbital diameters and let $r(x) < \infty$ for some $x \in M$. Then T has fixed point u in M and $u = \lim_{n \to \infty} T^n x$.*

Proof. Since $r(x)$ is finite, the sequence $\{r(T^n x)\}_{n=0}^{\infty}$ is a nonincreasing sequence of nonnegative reals. Let $\lim_{n \to \infty} r(T^n x) = \varepsilon \geqslant 0$, and assume that $\varepsilon > 0$. Since $r(T^n x)$ is finite for every $n = 0, 1, 2, \ldots$, by (1) we may choose a sequene

$$T^{n_1} x, T^{n_2} x, \ldots, T^{n_i} x, \ldots$$

in M such that

$$T^{n_i} x \in O(T^{n_{i-1}} x) \text{ and } r(T^{n_i} x) \leqslant \varphi[r(T^{n_{i-1}} x)] \tag{2}$$

$$(i = 1, 2, \ldots; \; n_0 = 0).$$

As $\{r(T^{n_i})\}_{i=1}^{\infty}$ is a subsequence of a convergent sequence $\{r(T^n x)\}_{n=0}^{\infty}$ it has the same limit, i.e.

(3) $$\lim_{i\to\infty} r(T^{n_i}x)=\varepsilon.$$

Since $\varepsilon\leqslant r(T^{n_i}x)$, by (2), (3) and definition of φ it follows

$$\varepsilon\leqslant\varphi(\varepsilon)<\varepsilon,$$

what is impossible. Consequently $\varepsilon=0$, what means that $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence. Since M is complete, there is $u\in M$ such that

$$\lim_{n\to\infty} T^n x=u.$$

Then $Tu=u$ by orbital continuity of T, and the proof is complete.

Note that one cannot delete a condition of orbital continuity of T in the theorem, even T satisfed the stronger condition:

(1′) $$r(Tx)\leqslant q\cdot r(x)$$

for some $q<1$, and M is compact. The following example shows it.

Example. Let

$$M=\{x:x=(\cos t,\ \sin t),\quad 0\leqslant t<2\pi\}$$

be a subset of Euclidean plane with usual metric. Define $T:M\to M$ by

$$T(t)=\frac{t+\pi}{2},\ \text{if}\ 0\leqslant t<\pi,$$

$$=\frac{t+2\pi}{2},\ \text{if}\ \pi\leqslant t<2\pi.$$

Then T satisfies (1′) with $q=2^{-\frac{1}{2}}$, but has not a fixed point.

Observe that mappings investigated by S. Husain and V. Sehgal in [3] and mappings investigated by S. Kasahara in [4] satisfy the condition (1). These mappings are orbitally continuous and may have only one fixed point.

REFERENCES

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