

LOGARITHMIC MEAN OF AN ENTIRE DIRICHLET SERIES

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1. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, where $s = \sigma + it$,

$$\lambda_{n+1} > \lambda_n, \lambda_1 \geq 0, \lim_{n \rightarrow \infty} \lambda_n = \infty \text{ and } \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = 0,$$

represent an entire Dirichlet series. The Ritt order ρ ($0 \leq \rho \leq \infty$) of $f(s)$ is defined [4, p. 78] as the limit superior of $\left\{ \frac{\log \log M(\sigma)}{\sigma} \right\}$, as $\sigma \rightarrow \infty$, with $M(\sigma) = \{|f(\sigma + it)| : -\infty < t < \infty\}$. The logarithmic mean of $f(s)$ is defined as:

$$(1.1) \quad L(\sigma) = \lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-T}^T \log |f(\sigma + it)| dt \right\}.$$

Since $\log L(\sigma)$ is an increasing convex function of σ [1, p. 13] we may write

$$(1.2) \quad \log L(\sigma) = \log L(x_0) + \int_{x_0}^{\sigma} U(x) dx, \quad \sigma \geq x_0,$$

where $U(\sigma)$ is a positive real valued indefinitely increasing function of σ (see [3], equation (4), p. 73).

Our aim in this paper is to study some of the growth properties of $U(\sigma)$ and $L(\sigma)$ relative to an auxiliary function. Various constants have been defined and a number of relations involving constants have been obtained.

2. For $0 < \rho \leq \infty$, let us set

$$(2.1) \quad \limsup_{\sigma \rightarrow \infty} \frac{U(\sigma)}{\inf_{e^{\rho\sigma}} \varphi(e^\sigma)} = \frac{a}{b}, \quad 0 < b \leq a < \infty,$$

and

$$(2.2) \quad \lim_{\sigma \rightarrow \infty} \frac{\log L(\sigma)}{\inf_{e^{\sigma}} \varphi(e^{\sigma})} = \frac{c}{d}, \quad 0 < d \leq c < \infty,$$

where $\varphi(e^{\sigma})$ satisfies the following two conditions:

- (i) $\varphi(x) > 0$ is continuous for $x > e^{\sigma_0}$, and
- (ii) $\varphi(lx) \approx \varphi(x)$, as $x \rightarrow \infty$, for every constant $l > 0$.

Now, we prove

Theorem 1. *The constants $a, b; c, d$ as defined by (2.1) and (2.2) satisfy the following relations*

$$(2.3) \quad b \leq \rho d \leq \rho c \leq a,$$

$$(2.4) \quad \rho c \geq \frac{a}{e} e^{b/a} \geq b,$$

$$(2.5) \quad \rho d \leq b \left\{ 1 + \log \left(\frac{a}{b} \right) \right\} \leq a,$$

and

$$(2.6) \quad a + \rho d \leq e \rho c.$$

Proof From (2.1), we have, for any $\varepsilon > 0$ and $\sigma \geq \sigma_0$,

$$(2.7) \quad (b - \varepsilon) e^{\sigma} \varphi(e^{\sigma}) < U(\sigma) < (a + \varepsilon) e^{\sigma} \varphi(e^{\sigma}).$$

Also, for $h < 0$, we have

$$(2.8) \quad \log L\left(\sigma + \frac{h}{\rho}\right) = \log L(\sigma_0) + \left\{ \int_{\sigma_0}^{\sigma} + \int_{\sigma}^{\sigma + \frac{h}{\rho}} \right\} U(x) dx.$$

Using left-hand inequality of (2.7) in (2.8), we get

$$\begin{aligned} \log L\left(\sigma + \frac{h}{\rho}\right) &> O(1) + (b - \varepsilon) \int_{\sigma_0}^{\sigma} e^{\sigma x} \varphi(e^x) dx + U(\sigma) \int_{\sigma}^{\sigma + \frac{h}{\rho}} dx \\ &\quad + O(1) + (b - E) \int_{\sigma}^{\sigma + \frac{h}{\rho}} x^{\beta-1} \varphi(x) dx + \frac{h U(\sigma)}{\rho}. \end{aligned}$$

Now

$$\int_{\sigma_0}^{\sigma} u^{\beta-1} \varphi(u) du \approx \frac{\sigma^{\beta} \varphi(\sigma)}{\beta},$$

for every positive β (see [2], lemma 5, p. 54), and so we get

$$\log L\left(\sigma + \frac{h}{\rho}\right) > O(1) + \frac{(b-\varepsilon)e^{\rho\sigma}\varphi(e^\sigma)}{\rho} + \frac{h}{\rho} U(\sigma).$$

Therefore

$$\frac{\log L\left(\sigma + \frac{h}{\rho}\right)}{e^{\rho\left(\sigma + \frac{h}{\rho}\right)} \varphi(e^\sigma)} > \frac{1}{e^h} \left\{ O(1) + \frac{b-\varepsilon}{\rho} + \frac{h}{\rho} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)} \right\}.$$

Taking limits and using (2.1) and (2.2) we get

$$(2.9) \quad c \geq \frac{1}{e^h} \left\{ \frac{b}{\rho} + \frac{a}{\rho} h \right\},$$

and

$$(2.10) \quad d \geq \frac{1}{e^h} \left\{ \frac{b}{\rho} + \frac{b}{\rho} h \right\}.$$

It can be seen that the maxima of the right-hand side expressions in (2.9) and (2.10) occur at $h = 1 - \frac{b}{a}$ and $h = 0$, respectively. Substituting, $h = 1 - \frac{b}{a}$ in (2.9) and $h = 0$ in (2.10), we get

$$(2.11) \quad \rho c \geq \frac{a}{e} e^{b/a} \geq b, \text{ and}$$

$$(2.12) \quad \rho d \geq b,$$

the last inequality in (2.11) follows from the fact that $e^x \geq ex$ for $x \geq 0$.

Again, on using right-hand inequality of (2.7) in (2.8), we obtain

$$\begin{aligned} \log L\left(\sigma + \frac{h}{\rho}\right) &\leq O(1) + (a+\varepsilon) \int_{\sigma_0}^{\sigma} e^{\rho x} \varphi(e^x) dx + U\left(\sigma + \frac{h}{\rho}\right) \int_{\sigma}^{\sigma + \frac{h}{\rho}} dx \\ &= O(1) + (a+\varepsilon) \int_{e^{\sigma_0}}^{e^{\sigma}} x^{\rho-1} \varphi(x) dx + \frac{h}{\rho} U\left(\sigma + \frac{h}{\rho}\right) \\ &\approx O(1) + \frac{(a+\varepsilon) e^{\rho\sigma} \varphi(e^\sigma)}{\rho} + \frac{h}{\rho} U\left(\sigma + \frac{h}{\rho}\right), \end{aligned}$$

and so we have

$$(2.13) \quad c \leq \frac{1}{e^h} \left\{ \frac{a}{\rho} + \frac{a}{\rho} h e^h \right\},$$

and

$$(2.14) \quad d \leq \frac{1}{e^h} \left\{ \frac{a}{\rho} + \frac{b}{\rho} h e^h \right\}.$$

It can also be seen that the minima of the right-hand side expressions in (2.13) and (2.14) occur at $h=0$ and $h=\log\left(\frac{a}{b}\right)$, respectively. Substituting $h=0$ in (2.13) and $h=\log\left(\frac{a}{b}\right)$ in (2.14), we get

$$(2.15) \quad \rho c \leq a, \text{ and}$$

$$(2.16) \quad \rho d \leq b \left\{ 1 + \log\left(\frac{a}{b}\right) \right\} \leq a,$$

since $1 + \log x \leq \exp(\log x)$.

Combining (2.12) and (2.15), since $\rho d \leq \rho c$, we get (2.3), (2.4) and (2.5) follow from (2.11) and (2.16), respectively.

To prove (2.6), we note that

$$U(\sigma) \leq \rho \int_{\sigma}^{\sigma + \frac{1}{\rho}} U(x) dx.$$

Adding $\rho \log L(\sigma)$ on both sides of the above inequality and using (1.2), we get

$$\rho \log L(\sigma) + U(\sigma) \leq \rho \log L\left(\sigma + \frac{1}{\rho}\right).$$

Dividing throughout by $e^{\rho\sigma} \varphi(e^\sigma)$ and taking limits, we get (2.6). This completes the proof of the theorem.

Remark: In the relation (2.4) actually $b < \frac{a}{e} e^{b/a}$ if $a \neq b$, and in the relation (2.5) $b \left\{ 1 + \log\left(\frac{a}{b}\right) \right\} < a$ if $a \neq b$. Thus the equality in the relations (2.4) and (2.5) will occur only if $a = b$. Moreover, from (2.4).

$$\frac{a}{e} e^{b/a} \leq \rho c$$

or,

$$(2.17) \quad a + b \leq e \rho c.$$

A comparison of (2.6) and (2.17) shows that (2.6) is a refinement of (2.17).

Theorem 2. If $f(s)$ is an entire function of Ritt order ρ ($0 < \rho < \infty$), then $\lim_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)}$ exists, if and only if, $\lim_{\sigma \rightarrow \infty} \frac{\log L(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)}$ exists, in which case

$$(2.18) \quad \lim_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)} = \rho \lim_{\sigma \rightarrow \infty} \frac{\log L(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)}.$$

Proof. If $\lim_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)}$ exists, then $\lim_{\sigma \rightarrow \infty} \frac{\log L(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)}$ exists follows from (2.3). We, therefore, suppose that

$$(2.19) \quad \lim_{\sigma \rightarrow \infty} \frac{\log L(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)} = c.$$

First let, $0 < c < \infty$, then for any $\varepsilon > 0$ and $\sigma \geq \sigma_0$, we have

$$(2.20) \quad (c - \varepsilon) e^{\rho\sigma} \varphi(e^\sigma) < \log L(\sigma) < (c + \varepsilon) e^{\rho\sigma} \varphi(e^\sigma).$$

Hence for any $1 > h > 0$, we get

$$\begin{aligned} h U(\sigma) &\leq \int_{\sigma}^{\sigma+h} U(x) dx \\ &= \int_0^{\sigma+h} U(x) dx - \int_0^{\sigma} U(x) dx \\ &= \log L(\sigma+h) - \log L(\sigma) \\ &< (c + \varepsilon) e^{\rho(\sigma+h)} \varphi(e^{\sigma+h}) - (c - \varepsilon) e^{\rho\sigma} \varphi(e^\sigma) \\ &= c(1 + \rho h + O(h^2))(1 + o(1)) e^{\rho\sigma} \varphi(e^\sigma) - c e^{\rho\sigma} \varphi(e^\sigma) \\ &\quad + o(e^{\rho\sigma} \varphi(e^\sigma)). \end{aligned}$$

Hence

$$\limsup_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)} \leq c(\rho + Hh),$$

where H is a constant. Since h is arbitrary and so making $h \rightarrow 0$, we find that

$$(2.21) \quad \limsup_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)} \leq \rho c.$$

By considering $\{\log L(\sigma) - \log L(\sigma-h)\}$ and proceeding as above, we get

$$(2.22) \quad \limsup_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)} \geq \rho c.$$

Thus, for $0 < c < \infty$, the relations (2.21) and (2.22) give us

$$(2.23) \quad \limsup_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{\rho\sigma} \varphi(e^\sigma)} = \rho c.$$

If $c=0$, then (2.21) gives $\lim_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{c\sigma} \varphi(e^\sigma)} = 0$ and if $c=\infty$, then taking an arbitrarily large number M in place of $c-\varepsilon$ and proceeding as above, we get $\lim_{\sigma \rightarrow \infty} \frac{U(\sigma)}{e^{c\sigma} \varphi(e^\sigma)} = \infty$.

The equality (2.18) follows from (2.19) and (2.23).

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