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# A NOTE ON CONSTRUCTIONS OF GRAPHS BY MEANS OF THEIR SPECTRA

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Abstract. The purpose of this note is to indicate by some examples the possibilities of constructing strongly regular graphs and block-designs using the theory of graph spectra.

In [2] and [3] we gave an idea as how to construct interesting graphs using the theory of graph spectra. One could start from graphs with known spectra and perform some graph operations on them in order to obtain graphs with the spectrum, which implies the graph properties we want.

A regular graph is strongly regular if the number of common neighbours of any two distinct vertices depends only on the fact whether they are adjacent or not.

In this note we shall construct some infinite families of strongly regular graphs and symmetric balanced incomplete block-designs (BIBD) starting from complete graphs and performing on them an $n$-ary operation[1] called NEPS [1].

Definition 1. Let $B$ be a set of $n$-tuples $(\beta_1, \ldots, \beta_n)$ of symbols 0 and 1, which does not contain $n$-tuple $(0, \ldots, 0)$. NEPS with the basis $B$ of the graphs $G_1, \ldots, G_n$ is the graph, whose vertex set is equal to the Cartesian product of the vertex sets of the graphs $G_1, \ldots, G_n$ and in which two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if there is an $n$-tuple $(\beta_1, \ldots, \beta_n)$ in $B$ such that $x_i = y_i$ holds exactly when $\beta_i = 0$ and $x_i$ is adjacent to $y_i$ in $G_i$ exactly when $\beta_i = 1$.

The complete graph $K_n$ on $n$ vertices has the spectrum consisting of eigenvalues $n-1$ and $-1$ with the multiplicities 1 and $n-1$, respectively. Next to complete graphs, according to the number of distinct eigenvalues, are strongly regular graphs. They have 3 distinct eigenvalues and any regular connected graph with 3 distinct eigenvalues is strongly regular.

The parameters of a strongly regular graph are the degree $r$, the number $e$ of common neighbours of any two adjacent vertices and the number $f$ of common neighbours af any two non-adjacent vertices. The distinct eigenvalues

[1] This operation appears also in a previous paper by the author and R. P. Lučić but it is due to the author (for details see [1]).

$\lambda_1, \lambda_2, \lambda_3$ $(\lambda_1>\lambda_2>\lambda_3)$ and the parameters are connected by the following relations

$$r=\lambda_1, \quad e=\lambda_1+\lambda_2+\lambda_3+\lambda_2\lambda_3, \; f=\lambda_1+\lambda_2\lambda_3.$$

Bipartite graphs with 4 distinct digenvalues are graphs on points and blocks of a symmetric BIBD, where the adjacency means the incidence of points and blocks, and again, the mentioned spectral property characterizes these graphs. If a symmetric BIBD has parameters $v, k, \lambda$, the corresponding eigenvalues are $\pm k$ and $\pm\sqrt{k-\lambda}$ with the multiplicities 1 and $v-1$, respectively [3].

The eigenvalues of NEPS can be expressed through the eigenvalues of the starting graphs [1]. If $\lambda_{ji_j}$ $(i_j=1, \ldots m_j)$ are the eigenvalues of $G_i$ $(i=1, \ldots, n)$, then the NEPS with the basis $B$ of graphs $G_1, \ldots, G_n$ has the eigenvalues $\Lambda_{i_1, \ldots, i_n}$ $(i_j=1, \ldots, m_j, \; j=1, \ldots, n)$, where

$$\Lambda_{i_1, \ldots, i_n}=\sum_{\beta\in B}\lambda_{1i_1}^{\beta_1}\ldots\lambda_{ni_n}^{\beta_n} \tag{1}$$

and $\beta=(\beta_1, \ldots, \beta_n)$.

In virtue of (1), the number of distinct eigenvalues of a NEPS can be expected to be much greater than the number of distinct eigenvalues of starting graphs. In exceptional cases it turns out to be possible that a NEPS of complete graphs contains only 3 or 4 distinct eigenvalues which leads to the construction of strongly regular graphs or possibly of symmetric BIBD's. We introduce now some special cases of NEPS, which will be useful in the constructions.

Definition 2. The odd (even) sum of graphs is a NEPS with the basis containing all the $n$-tuples with an odd (even) number of 1's.

Definition 3. The mixed sum of graphs is a NEPS with the basis containing all the $n$-tuples in which the number of 1's is congruent to 1 or 2 modulo 4.

Example 1. The odd sum of two graphs $K_n$ is a strongly regular graph with the eigenvalues $2n-2, n-2, -2$. It is isomorphic to the line graph $L(K_{n,n})$ of a complete bipartite graph $K_{n,n}$.

In Theorems 1 and 2 we construct two more infinite series of strongly regular graphs by means of the NEPS.

Theorem 1. *The odd sum $F_n$ of $n$ $(n\geqslant 2)$ copies of the graph $K_4$ is a strongly regular graph with the eigenvalues* $2^{2n-1}+(-1)^{n-1}2^{n-1}, \; 2^{n-1}, \; -2^{n-1}$.

Proof. The distinct eigenvalues of $K_4$ are 3 and $-1$. Let $S_p$ be the elementary symmetric function of order $p$ of variables $x_1, \ldots, x_n$ and let these variables take the values 3 or $-1$. If $k$ variables take value 3 and if the remaining $n-k$ ones take value $-1$, the value of $S_i$ is equal to the coefficient of $x^{n-i}$ in the polynomial $P_k(x)=(x-3)^k(x+1)^{n-k}$. According to (1), the eigenvalues of the odd sum are given by $\sum_i S_i$ where the summation goes over all odd numbers $i$ not greater than $n$. If $P_k(x)=a_0x^n+a_1x^{n-1}+\cdots+$ $+a_{n-1}x+a_n$, we have

$$\Lambda_k \overset{\text{def}}{=} -(a_1+a_3+\cdots)=\frac{1}{2}((-1)^nP_k(-1)-P_k(1)), \quad k=0, 1, \ldots, n.$$

$\Lambda_k$ are the eigenvalues of $F_n$ and we immediately get $\Lambda_k=(-1)^{k+1}2^{n-1}$ $(k=0, 1, \ldots, n-1)$ and $\Lambda_n=2^{2n-1}+(-1)^{n-1}2^{n-1}$, which proves the theorem.

$F_n$ is a regular graph of degree $2^{2n-1}+(-1)^{n-1}2^{n-1}$ on $4^n$ vertices. Any two distinct vertices have $2^{2n-2}+(-1)^{n-1}2^{n-1}$ common neighbours. $F_n$ can be visualized in the following way. The $4^n$ $n$-tuples of 4 distinct symbols are the vertices and two $n$-tuples are adjacent if they differ in an odd number of coordinates.

The even sum of graphs $F_n$ and $K_2$ is a regular bipartite graph with 4 distinct eigenvalues $\pm(2^{2n-1}+(-1)^{n-1}2^{n-1})$, $\pm 2^{n-1}$. Hence, we have constructed a symmetric BIBD with the parameters

$$v=b=4^n,\ r=k=2^{2n-1}+(-1)^{n-1}2^{n-1},\ \lambda=2^{2n-2}+(-1)^{n-1}2^{n-1}.$$

The Seidel spectrum of $F_n$ consists of only two numbers, $2^n-1$ and $-2^n-1$, and the corresponding switching class represents a regular two-graph [6]. The complement of symplectic two-graph has the same eigenvalues [5], [6] and, according to [4], the two two-graphs coincide.

The graphs $F_n$ and the corresponding block-designs have been constructed in the literature in many different ways (cf., e.g., [4]) and the construction given above is still a new one.

Theorem 2. *The mixed sum $H_s$ of $4s\,(s\geqslant 1)$ copies of the graph $K_2$ is a strongly regular graph with the eigenvalues $2^{4s-1}-(-1)^s2^{2s-1}$, $2^{2s-1}$, $-2^{2s-1}$.*

Proof. Eigenvalues of $K_2$ are 1 and $-1$. In order to obtain the eigenvalues of the mixed sum of $n=4s$ copies of $K_2$ consider the polynomial

$$Q_k(x)=(x-1)^k(x+1)^{n-k}=b_0x^n+b_1x^{n-1}+\cdots+b_{n-1}x+b_n. \tag{2}$$

Define $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ by the formulas

$$\alpha_0=b_0+b_4+b_8+\cdots+b_n, \qquad \alpha_1=b_1+b_5+b_9+\cdots+b_{n-1},$$
$$\alpha_2=b_2+b_6+b_{10}+\cdots+b_{n-2}, \qquad \alpha_3=b_3+b_7+b_{11}+\cdots+b_{n-3}.$$

The values of $-\alpha_1+\alpha_2$ for $k=0, 1, \ldots, n$ represent the eigenvalues of $H_s$. First we have

$$Q_k(1)=\alpha_0+\alpha_1+\alpha_2+\alpha_3, \qquad Q_k(-1)=\alpha_0-\alpha_1+\alpha_2-\alpha_3,$$
$$Q_k(i)=\alpha_0-i\alpha_1-\alpha_2+i\alpha_3, \qquad Q_k(-i)=\alpha_0+i\alpha_1-\alpha_2-i\alpha_3.$$

Further we obtain

$$-\alpha_1+\alpha_2=\frac{1}{4}(2Q_k(-1)-(1+i)Q_k(i)-(1-i)Q_k(-i))\stackrel{\text{def}}{=}\Lambda_k.$$

Using (2) and (3) we get $\Lambda_k=(-1)^{s-1+\binom{k}{2}}2^{2s-1}$ $(k=0, 1, \ldots, 4s-1)$ and $\Lambda_{4s}=2^{4s-1}-(-1)^s2^{2s-1}$, which proves the theorem.

The Seidel spectrum of $H_s$ contains the eigenvalues $2^{2s}-1$ and $-2^{2s}-1$ and we have again a regular two-graph with the same eigenvalues as with $F_{2s}$ of Theorem 1.

$H_s$ has $16^s$ vertices, degree $2^{4s-1}-(-1)^s 2^{2s-1}$, and any two distinct vertices have $2^{4s-2}-(-1)^s 2^{2s-1}$ common neighbours. $H_1$ is the well-known Clebsh graph on 16 vertices with the eigenvalues 10, 2, $-2$. $H_s$ can be realized as the graph whose vertices are $4s$-tuples of two symbols, two vertices being adjacent if the number of coordinates in which they differ is congruent to 1 or 2 modulo 4.

The even sum of $H_s$ and $K_2$ again gives a block-design graph. The eigenvalues are $\pm(2^{4s-1}-(-1)^s 2^{2s-1})$, $\pm 2^{2s-1}$, and this corresponds to a BIBD with parameters $v=b=16^s$, $r=k=2^{4s-1}-(-1)^s 2^{2s-1}$, $\lambda=2^{4s-2}-(-1)^s 2^{2s-1}$.

Graphs $F_{4p}$ and $H_{2p}$ have the same spectrum and it would be interesting to know whether they are isomorphic. If yes, then the graph equation $O_{4p}(G)=$ $=M_{2p}(H)$, where $O_n(G)$ and $M_n(G)$ denote the odd and the mixed sum of $n$ copies of the graph $G$, has a solution $G=K_4$, $H=K_2$. Are there any other solutions?

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## REFERENCES

[1] D. M. Cvetković, *Graphs and their spectra,* Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № 354—№ 356 (1971), 1—50.

[2] D. M. Cvetković, *Some possible directions in further investigations of graph spectra,* Proc. Internat. Coll. on Algebraic Methods in Graph Theory, Szeged, August 25—31, 1978, to appear.

[3] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of graphs,* Theory and application, Deutscher Verlag der Wissenschaften — Academic Press, Berlin—New York. 1980.

[4] W. M. Kantor, *Symplectic groups, symmetric design and line ovals,* J. of Algebra 33 (1975), 43—58.

[5] J. J. Seidel, *On two-graphs and Shult's characterization of symplectic and orthogonal geometries over GF(2),* Technological Univ. Eindhoven, T. H. — Report 73—WSK—02.

[6] J. J. Seidel, *A survey of two-graphs,* Proc. Int. Coll. Theorie Combinatorie, Acc. Naz. Lincei, Roma 1973, Roma 1976, 481—511.