

CHARACTERISTIC OPERATOR FUNCTIONS ON WACHS SPACES

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Abstract. — In this note, the notions of operator knots on Wachs spaces (quaternionic Hilbert spaces) and the corresponding characteristic operator functions are considered. We obtained the generalizations of some basic properties of these notions in Hilbert spaces, following the first chapter of the M. S. Brodskii's monography [6] (pp. 7—102).

We do not quote many other results concerning invariant subspaces and characteristic operator functions, because they are in results and proofs direct translations from complex Hilbert spaces.

1. Operator knots in Wachs spaces.

Let H, G be two separable Wachs spaces, i.e. quaternionic Hilbert spaces,

$$A \in L(H), K \in L(G, H), J \in L(G),$$

be bounded linear operators on these spaces such that

$$J^* = -J, J^2 = -I.$$

Let next H^s, G^s be simplectic images of spaces H, G , and let B^s denote the simplectic image of an arbitrary bounded linear operator on H or G . Then

$$\overline{B^s} = SB^s(-S) = B^s$$

where $S = jI$ (i, j, k -are quaternionic units).

If it holds

$$KJK^* = \text{Im}(A) = \frac{1}{2}(A - A^*)$$

then

$$(*) \quad \theta = \begin{pmatrix} A & K & J \\ H & & G \end{pmatrix}$$

is an operator knot, and

$$\theta^s = \begin{pmatrix} A^s & K^s & J \\ H^s & & G^s \end{pmatrix}$$

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where $J = -iJ^s$, is its symplectic image, thus an operator knot in complex Hilbert spaces.

We only emphasize that

$$\bar{J} = -J \quad (\text{in } G^s).$$

Then:

- H — is the exterior space,
- G — interior space,
- A — basic operator,
- K — canal operator,
- J — directional operator of the knot θ .

As in the complex Hilbert spaces, every operator $A \in L(H)$ can be „embedded“ into a operator knot θ , whose basic operator is A .

Theorem 1. — *Let A be a bounded linear operator on a separable Wachs space H , and E by any closed subspace containing the range $R(\text{Im } A)$. Then there is an operator knot θ whose basic operator is A .*

Proof. Since the subspace $E \supseteq R(\text{Im } A)$, its symplectic image E^s in H^s contains the complex subspace

$$R\left(\frac{A^s - A^{s*}}{2}\right) = iR(\text{Im } A^s) = R(\text{Im } A^s).$$

Besides, it is easily check that the spectrum of $A_I = \text{Im}(A^s)$ is symmetric with respect to 0, since it is selfadjoint and

$$\overline{A_I} = (-i \overline{\text{Im } A^s}) = -A_I.$$

Let us denote: $A_I^{(0)} = A_I|_{\overline{R(A_I)}}$ and $E^s(t)$ — the spectral function of the operator $A_I^{(0)}$ in $\overline{R(A_I)}$. Then:

$$A_I^{(0)} = \int_{-a}^a t dE^s(t) \quad (a \geq 0).$$

Since next:

$$A_I^{(0)} = -\overline{A_I^{(0)}} = - \int_{-a}^a t dE^s(t) = \int_{-a}^a t d(I - \overline{E^s(-t)})$$

and $F^s(t) = I - \overline{E^s(-t)}$ ($-a \leq t \leq a$) is an orthogonal decomposition of unit on H^s , it follows:

$$(1.1) \quad E^s(t) \equiv I - \overline{E^s(-t)} \quad (-a \leq t \leq a).$$

If we put

$$K_0 = \int_{-a}^a |t|^{1/2} dE^s(t),$$

$$J_0 = \int_{-a}^a \text{sgnt } dE(t),$$

then

$$J_0^* = J_0, \quad J_0^2 = I,$$

$$K_0 J_0 K_0^* = A_I^{(0)} \quad (\text{in } \overline{R(A_I)})$$

([6], p. 12).

In view of (1.1), it is then easily seen that

$$\begin{aligned} \overline{K_0} &= \int_{-a}^a \sqrt{|t|} d\overline{E^s(t)} = \int_{-a}^a \sqrt{|t|} d(I - \overline{E^s(-t)}) = \\ &= \int_{-a}^a \sqrt{|t|} dE^s(t) = K_0, \end{aligned}$$

and $\overline{J_0} = -J_0$.

The rest of the proof is then very similar to the corresponding proof of Theorem 1.1 ([6], p. 12).

Hence there is a Wachs space G , an $K^s \in L(G, H)$ ($\overline{K^s} = K^s$), and a selfadjoint operator $J^s \in L(G^s)$ such that $(J^s)^2 = I$, $\overline{J^s} = -J^s$ and

$$K^s J^s K^{s*} = \frac{A^s - A^{s*}}{2i}.$$

Now defining

$$Kx = K^s x, \quad Jx = iJ^s x \quad (x \in G^s),$$

we get a desired operator knot. \square

— A Wachs space operator knot θ is said to be *simple*, if the linear hull of sets $A^n R(K)$ ($n = 0, 1, 2, \dots$) is dense in H .

Since $A^n R(K)$ are subspaces of the space H , we see that θ is a simple knot, if and only if its symplectic image θ^s is a complex simple knot.

2. Characteristic operator function.

If $\theta = \begin{pmatrix} A & K & J \\ H & & G \end{pmatrix}$ is an operator knot, we put:

$$W_\theta(\lambda) = W_{\theta^s}(\lambda) \quad (\lambda \in \rho(A)),$$

and, as usually, we call $W_\theta(\lambda)$ the characteristic operator function related to the knot θ .

It is defined and holomorphic on the resolvent set $\rho(A)$ of the operator A , which is symmetric about the real axis: $\rho(A) = \rho(A)^*$.

Thus

$$W_\theta(\lambda) = I - 2i K^{s*} R_{A^s}(\lambda) K^s J = I - 2i K^{s*} (A^s - \lambda I)^{-1} K^s J \quad (\lambda \in \rho(A))$$

defines a family of bounded linear operators on the space G^s .

Together with the usual properties of $W_\theta(\lambda)$, we immediately get

$$(2.1) \quad W_\theta(\overline{\lambda}) \equiv \overline{W_\theta(\lambda)} \quad (\lambda \in \rho(A)).$$

Indeed, for an arbitrary $\lambda \in \rho(A)$,

$$\begin{aligned}\overline{W_\theta(\bar{\lambda})} &= S[I - 2i K^{s*} (A^s - \bar{\lambda} I)^{-1} K^s J] \bar{S} = \\ &= I + 2i \overline{K^{s*}} (\overline{A^s - \lambda I})^{-1} \overline{K^s J} = \\ &= I - 2i K^{s*} (A^s - \lambda I)^{-1} K^s J = \\ &= W_\theta(\lambda). \square\end{aligned}$$

If next $\overline{K^s} = \overline{K^s}$, $\overline{J^s} = -J$, and $\overline{A^s} \neq A^s$, then it is interesting to see which occurs if (2.1) holds true.

Then obviously $\text{Im}(A) \in L(H)$, so that

$$\overline{\text{Im}(A^s)} = -\text{Im}(A^s).$$

Theorem 2. — Let θ^s be a simple knot with basic operator $A^s \in L(H^s)$, let $\rho(A^s)$ be symmetric about the real axis, and

$$W_{\theta^s}(\bar{\lambda}) \equiv \overline{W_{\theta^s}(\lambda)} \quad (\lambda \in \rho(A^s)).$$

Then operators $\overline{A^s}$ and A^s must be unitarily equivalent in G^s . Moreover, it holds:

$$\begin{aligned}\overline{A^s} x &= A^s x \quad (x \in \overline{R(K^s)}), \\ \overline{A^s} y &= U_0 A^s U_0^{-1} y \quad (y \in \overline{R(K^s)}^\perp),\end{aligned}$$

for an unitar operator U_0 in the subspace $\overline{R(K^s)}^\perp$.

Proof. In view of Theorem 3.2 ([6], p. 26), there is an unitar operator U^s on H^s such that

$$\overline{A^s} = U^s A^s (U^s)^{-1}, \quad K^s = U^s K^s.$$

Then $U^s x = x$ ($x \in \overline{R(K^s)}$), and implicitly, restriction U_0 of U^s on the subspace $\overline{R(K^s)}$ is an unitar operator on this subspace. Therefrom we get statement. \square

Remark. It is easy to see that if $\overline{K^s} = K^s$, $\overline{J^s} = -J$, then $\overline{W_{\theta^s}(\bar{\lambda})}$ is the characteristic operator function of the knot

$$\theta^s = \begin{pmatrix} \overline{A^s} & K^s & J \\ H^s & & G^s \end{pmatrix}. \square$$

Next, let us put:

$$T(\lambda) = W_\theta^*(\lambda) J W_\theta(\lambda) - J \quad (\theta\text{-a knot}).$$

Then $T(\lambda)$ is defined and holomorphic operator function on the set $\rho(A)$ with values in $L(G^s)$. Since $\lambda \in \rho(A)$ implies $\bar{\lambda} \in \rho(A)$, in view of relation 3.12 ([6], p. 29) we get:

$$(2.2) \quad W_\theta(\lambda)^{-1} = J W_\theta^*(\lambda) J \quad (\lambda \in \rho(A)).$$

Next it holds true:

$$(2.3) \quad T(\lambda) \geq 0 \quad (\operatorname{Im} \lambda > 0, \lambda \in \rho(A)),$$

$$(2.4) \quad T(\lambda) \leq 0 \quad (\operatorname{Im} \lambda < 0, \lambda \in \rho(A)),$$

$$(2.5) \quad T(\bar{\lambda}) \equiv -T(\lambda) \quad (\lambda \in \rho(A)).$$

If now:

$$V_0(\lambda) = K^* (\operatorname{Re}(A^s) - \lambda I)^{-1} K \quad (\lambda \in \rho(\operatorname{Re}(A))),$$

then $V_0(\lambda)$ is defined and holomorphic operator function on $\rho(\operatorname{Re} A)$ with values in $L(G^s)$ such that

$$(2.6) \quad \operatorname{Im} V_0(\lambda) \geq 0 \quad (\operatorname{Im} \lambda > 0, \lambda \in \rho(\operatorname{Re} A)),$$

$$(2.7) \quad \operatorname{Im} V_0(\lambda) \leq 0 \quad (\operatorname{Im} \lambda < 0, \lambda \in \rho(\operatorname{Re} A)),$$

$$(2.8) \quad V_0(\bar{\lambda}) \equiv \overline{V_0(\lambda)} \quad (\lambda \in \rho(\operatorname{Re} A))$$

hold true.

We emphasize that (2.7) is a consequence of (2.6) and (2.8). Besides, as it is known:

$$V_0(\lambda) = (W_0(\lambda) + I)^{-1} (W_0(\lambda) - I) iJ,$$

$$W_0(\lambda) = (I + V_0(\lambda) iJ)^{-1} (I - V_0(\lambda) iJ)$$

$$(\lambda \in \rho(A) \cap \rho(\operatorname{Re} A)).$$

Lemma 1. *An operator function $V(\lambda) \in L(G^s)$ can be put into the form*

$$(*) \quad V(\lambda) = \int_a^b \frac{dF(t)}{t - \lambda} \quad (\lambda \in C \setminus [a, b]),$$

where $\overline{F(t)} \equiv F(t)$ ($a \leq t \leq b$) is a non negative non decreasing operator function on G^s , iff it holds:

$$(1^\circ) \quad V(\lambda) \text{ holomorphis in } C \setminus [a, b];$$

$$(2^\circ) \quad V(\infty) = 0;$$

$$(3^\circ) \quad \operatorname{Im} V(\lambda) \geq 0 \quad (\operatorname{Im} \lambda > 0);$$

$$(4^\circ) \quad \operatorname{Im} V(\lambda) = 0 \quad (\lambda \in R \setminus [a, b]);$$

$$(5^\circ) \quad V(\bar{\lambda}) \equiv \overline{V(\lambda)} \quad (\lambda \in C \setminus [a, b]).$$

Proof. Relations $(1^\circ) - (5^\circ)$ easily follows from the decomposition (*). Conversely, by the Theorem 4.9 ([6], p. 41), using $(1^\circ) - (4^\circ)$ it follows

$$V(\lambda) = \int_a^b \frac{dF(t)}{t - \lambda} \quad (\lambda \in C \setminus [a, b]).$$

In view of relation (5°) , we have

$$\int_a^b \frac{d\overline{F(t)}}{t - \lambda} = \int_a^b \frac{dF(t)}{t - \lambda} \quad (\lambda \notin [a, b]).$$

We can suppose that

$$F(a) = F(a-0) = 0, \quad F(b+0) = F(b).$$

Since $\overline{F(t)}$ is non negative and non decreasing operator function ($a \leq t \leq b$) such that

$$\overline{F(a)} = \overline{F(a-0)} = 0, \quad \overline{F(b+0)} = \overline{F(b)}$$

from the uniqueness theorem we have that

$$\overline{F(t)} \equiv F(t) + C \quad (a \leq t \leq b)$$

thus $C = \overline{F(a)} - F(a) = 0$. Consequently,

$$\overline{F(t)} \equiv F(t) \quad (a \leq t \leq b). \quad \square$$

3. The class $\Omega_J(Q)$.

Let an operator $J \in L(G^s)$ be such that $J^* = J = J^{-1}$ and $\bar{J} = -J$. We recall that an operator function $W(\lambda) \in L(G^s)$ belongs to the class $\Omega_J = \Omega_J(C)$ iff:

(I) $W(\lambda)$ is holomorphic in a neighbourhood G_w of $z = \infty$;

(II) $\|W(\lambda) - I\| \rightarrow 0 \quad (\lambda \rightarrow \infty)$;

(III) $(W(\lambda) + I)^{-1}$ exists for every $\lambda \in G_w$, and

$$V(\lambda) = (W(\lambda) + I)^{-1} (W(\lambda) - I) (iJ) = (W(\lambda) - I) (W(\lambda) + I)^{-1} (iJ)$$

can be analytically extended in a region $G_v = C \setminus [a, b]$ ($-\infty < a < b < +\infty$);

(IV) $\operatorname{Im} V(\lambda) \geq 0 \quad (\operatorname{Im} \lambda > 0)$;

(V) $\operatorname{Im} V(\lambda) = 0 \quad (\lambda \in R \setminus [a, b])$

([6], p. 42).

We define:

$\Omega_J(Q)$ is the class of all $W(\lambda) \in \Omega_J(C)$ such that $G_w = (G_w)^*$, and

(VI) $W(\bar{\lambda}) \equiv \overline{W(\lambda)} \quad (\lambda \in G_w)$

holds true. \square

Proposition 1. Characteristic operator function $W_0(\lambda)$ of an arbitrary knot

$$\theta = \begin{pmatrix} A & K & J \\ H & & G \end{pmatrix}$$

belongs to the class $\Omega_J(Q)$. \square

Theorem 3. *If an operator function $W(\lambda) \in \Omega_J(Q)$, then there is a knot θ with directional operator J , such that for an $l > 0$*

$$W_\theta(\lambda) = W(\lambda) \quad (|\lambda| \geq l).$$

Proof. By virtue of Lemma 1, proof of the theorem is quite analogous to the corresponding proof of Theorem 5.1 ([6], p. 43). We only emphasize that the generalized Naimark's theorem holds true in Wachs spaces also. \square

Theorem 3a. *If an operator function $\bar{W}(\lambda) \in \Omega_J(Q)$ then there is a simple operator knot θ whose directional operator is J , such that for an $l > 0$*

$$W_\theta(\lambda) \equiv W(\lambda) \quad (|\lambda| \geq l). \square$$

4. Finite dimensional knots.

An operator knot θ is said to be finite dimensional, if θ^* is so, thus iff the spaces H and G are finite dimensional.

If next $J^* = -J$, $J^2 = -I$ in a m -dimensional Wachs space $G = G_m (m \in \mathbb{N})$, let $\Omega_J(F; Q)$ be the class of all operator functions

$$W(\lambda) \in \Omega_J(F) = \Omega_J(F; C) \quad (J = iJ^*)$$

related to the spaces G_{2m}^s , such that

$$W(\bar{\lambda}) \equiv \overline{W(\lambda)}, \quad \lambda \in D_W \cap (D_W)^* \quad ([6], \text{ p. } 83).$$

Then, every c. o. f. $W_\theta(\lambda)$ of a finite dimensional knot θ belongs to $\Omega_J(F; Q)$, and conversely — every $W(\lambda) \in \Omega_J(F; Q)$ is the c. o. f. of some simple finite dimensional knot θ .

Theorem 4. *An operator function $W(\lambda) \in \Omega_J(F; Q)$ iff it holds*

$$(4.1) \quad W(\lambda) \equiv W_0(\lambda) \overline{W_0(\bar{\lambda})} \quad (\lambda \in D_{W_0} \cap (D_{W_0})^*),$$

where $W_0(\lambda) \in \Omega_J(F)$.

Proof. (a) — If at first $W_0(\lambda) \in \Omega_J(F)$ then by Theorem 11.2 ([6], p. 85)

$$W_0(\lambda) = \prod_{r=1}^{2n} \left(I + \frac{2i\sigma_r}{\lambda - \lambda_r} P_r J \right)$$

where $\lambda_1, \dots, \lambda_{2n}$ are complex numbers, $\sigma_1, \dots, \sigma_{2n}$ are positive numbers and P_1, \dots, P_{2n} are one-dimensional projections in some G_{2n}^s , such that

$$P_r J^s P_r = \frac{\operatorname{Im} \lambda_r}{\sigma_r} P_r \quad (r = 1, \dots, 2n).$$

Then

$$\overline{W_0(\bar{\lambda})} = \prod_{r=1}^{2n} \left(I + \frac{2i\sigma_r}{\lambda - \lambda_r} \bar{P}_r J \right),$$

but since $\bar{P}_1, \dots, \bar{P}_{2n}$ are one-dimensional projections too such that

$$\bar{P}_r J \bar{P}_r = -\overline{P_r J P_r} = -\frac{\operatorname{Im} \lambda_r}{\sigma_r} \bar{P}_r = \frac{\operatorname{Im} \bar{\lambda}_r}{\sigma_r} \bar{P}_r,$$

it follows that operator function $\overline{W_0(\bar{\lambda})} \in \Omega_J(F)$ also. Consequently,

$$W(\lambda) \stackrel{\text{def}}{=} W_0(\lambda) \overline{W_0(\bar{\lambda})} \in \Omega_J(F; Q)$$

because $W(\bar{\lambda}) \equiv \overline{W(\lambda)}$ ($\lambda \neq \lambda_1, \bar{\lambda}_1, \dots, \lambda_{2n}, \bar{\lambda}_{2n}$).

(b) — It remains to prove that factorisation (4.1) holds true for every $W(\lambda) \in \Omega_J(F; Q)$.

At first we have that

$$W(\lambda) \equiv W_\theta(\lambda)$$

for a finite dimensional knot $\theta = \begin{pmatrix} A & K & J \\ H_n & G_m \end{pmatrix}$.

Exactly as in Lemma 11.1 ([6], p. 84), we prove that in space $H = H_n$ there is an orthonormal basis e_1, \dots, e_n such that

$$(*) \quad Ae_r = q_{r1} e_1 + \dots + q_{r, r-1} e_{r-1} + \lambda_r e_r \quad (r = 1, \dots, n)$$

where $\lambda_1, \dots, \lambda_n \in C$ and q_{rs} are quaternions ($r, s = 1, \dots, n; r > s$).

If $q_{rs} = a_{rs} + b_{rs}j$ ($a_{rs}, b_{rs} \in C$) then

$$Ae_r = (a_{r1} e_1 + b_{r1} j e_1) + \dots + (a_{r, r-1} e_{r-1} + b_{r, r-1} j e_{r-1}) + \lambda_r e_r$$

($r = 1, \dots, n$), so that

$$A(je_r) = (-\bar{b}_{r1} e_1 + \bar{a}_{r1} j e_1) + \dots + (-\bar{b}_{r, r-1} e_{r-1} + \bar{a}_{r, r-1} j e_{r-1}) + \bar{\lambda}_r j e_r.$$

Now put:

$$f_{2r-1} = e_r, f_{2r} = j e_r \quad (r = 1, \dots, n).$$

So we obtain an orthonormal basis f_1, f_2, \dots, f_{2n} in the space H_{2n}^s such that

$$f_{2r} = j f_{2r-1} \quad (r = 1, \dots, n)$$

Hence, in the basis f_1, \dots, f_{2n} , matrix of the operator A^s must be upper triangular and

$$[A^s] = \begin{bmatrix} \lambda_1 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & \bar{\lambda}_1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \lambda_n & 0 \\ 0 & 0 & \cdot & \cdot & 0 & \bar{\lambda}_n \end{bmatrix}$$

Let now H_r^s be the linear hull of f_1, \dots, f_r in the space H_{2n}^s ($r=1, \dots, 2n$). Then H_1^s, \dots, H_{2n}^s are invariant subspaces of A^s and

$$O = H_0^s \subseteq H_1^s \subseteq H_2^s \subseteq \dots \subseteq H_{2n}^s$$

($\dim H_r^s = r$). Further, put $M_r = H_r^s \ominus^s H_{r-1}^s$, i.e. $M_r = L\{f_r\}$. Then

$$M_{2r} = SM_{2r-1} \quad (r=1, \dots, n),$$

and

$$M_1 \oplus^s M_2 \oplus^s \dots \oplus^s M_{2n-1} \oplus M_{2n} = H_{2n}^s.$$

If P_r is the orthogonal projection of H_{2n}^s on M_r ($r=1, \dots, 2n$), then

$$\bar{P}_{2r-1} = P_{2r} \quad (r=1, \dots, n).$$

Now if

$$\theta_r = \text{pr } M_r(\theta) \quad (r=1, \dots, 2n)$$

then easily,

$$W_{\theta_{2r-1}}(\lambda) = I + \frac{2\sigma_r}{\lambda - \lambda_r} P_{2r-1}(iJ),$$

and

$$W_{\theta_{2r}}(\lambda) = I + \frac{2\sigma_r}{\lambda - \lambda_r} P_{2r}(iJ) = I + \frac{2\sigma_r}{\lambda - \lambda_r} \bar{P}_{2r-1}(\bar{iJ}),$$

thus

$$W_{\theta_{2r}}(\lambda) \equiv \overline{W_{\theta_{2r-1}}(\bar{\lambda})} \quad (\lambda \neq \lambda_r, \lambda_r).$$

Since

$$W(\lambda) = \prod_{r=1}^{2n} W_{\theta_r}(\lambda) = \prod_{r=1}^n W_{\theta_{2r-1}}(\lambda) \prod_{r=1}^n W_{\theta_{2r}}(\lambda) = \prod_{r=1}^n W_{\theta_{2r-1}}(\lambda) \prod_{r=1}^n \overline{W_{\theta_{2r-1}}(\bar{\lambda})}$$

($\lambda \neq \lambda_1, \bar{\lambda}_1, \dots, \lambda_n, \bar{\lambda}_n$), we obtain that

$$W(\lambda) \equiv W_0(\lambda) \overline{W_0(\bar{\lambda})}$$

where obviously

$$W_0(\lambda) = \prod_{r=1}^n W_{\theta_{2r-1}}(\lambda) \in \Omega_J(F).$$

This finishes the proof. \square

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