PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 27 (41) 1980, pp. 267—277

VERY STRONGLY RIGID BOOLEAN ALGEBRAS

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(Received October 19, 1979)

0. Introduction and definitions

The present work is a continuation of our work [*T*]. Namely, it happened that on the problematics of rigid Boolean algebras worked independently at the same time other mathematicians as well. As a good review of what has been recently done in connection with fundamental set-theoretic problems concerning Boolean algebras one has [vDMR]. In some recent papers appeared problems which explicitly or implicitly are resolved in our paper [*T*] (see [Bo 1, Problems 3,4 and 7], [Bo 3; Problème 3], [vDMR; (4), (14) and Question 16], [Lo; Problem 6.17], [LR; p 347]). Therefore the aim of the present paper is to show it with more details. Besides new results are added also.

Boolean algebra (*BA*) is called *rigid* if it has no non-trivial automorphisms. For arbitrary *BA*'s one can consider various strengthenings of the notion of rigidity (see, e. g., [vDMR], [Bo 2], [Lo]):

B is *mono-rigid* if every one-to-one endomorphism is the identity.

B is *onto-rigid* if every onto endomorphism is the identity.

B is *bi-rigid* if it is both mono — and onto-rigid.

B is *very strongly rigid* if for every *BA* B', every one-to-one homomorphism F from B into B' and every homomorphism G from B onto B' we have $F=G$.

It is easy to see that every very strongly rigid *BA* is bi-rigid, that every bi-rigid *BA* is mono — and onto-rigid and that every mono — or onto-rigid *BA* is rigid.

A majority of *BA*'s we shall construct in this paper are *Boolean algebras with ordered bases*. A *BA* with an ordered base can be realised as the set of all finite unions of intervals, of a linearly ordered set; and conversely. So that we shall refer to algebras with ordered bases as *interval algebras* and use the following notation. If L is a linearly ordered set then by $B(L)$ we shall mean the set of all finite unions of intervals of L of the form $[x, y)$, $x, y \in L \cup \{\pm\infty\}$. Call $B(L)$ the *interval algebra on L*.

Call a *BA B retractive* if for every onto homomorphism $G: B \to B'$ there exists a one-to-one homomorphism $F: B' \to B$ such that GF is the identity on the *BA* B' (i. e. $GF = id_{B'}$).

Proposition 0.1. *If B is mono-rigid subalgebra of an interval algebra, then B is very strongly rigid.*

Proof. Assume there are homomorphisms $F: B \to B'$ and $G: B \to B'$ such that F is one-to-one and G is onto. Since every subalgebra of an interval algebra is retractive (see [*R*] and [Ro; Theorem 1.5]) there is a one-to-one homomorphism $H: B' \to B$ such that $GH = id_{B'}$. Now HF is the identity on B since it is a one-to-one endomorphism of B, a mono rigid *BA*. So H is onto and hence G is one-to-one. Now $G^{-1}F$ is the identity on B. So, $F = G$.

If λ is a limit ordinal, then $C \subseteq \lambda$ is closed and unbounded in λ (club in λ) if $\sup C = \lambda$ and $\sup(C \cap \alpha) \in C$ or $C \cap \alpha = \varnothing$ for all $\alpha < \lambda$. S is *stationary in* λ if $S \cap C \neq \varnothing$ for all club C in λ. Let $D(\lambda)$ be the *BA* $P(\lambda)/I(\lambda)$, where $P(\lambda)$ is the power set of λ and $I(\lambda)$ is the ideal of all subsets of λ that are disjoint from some club subset of λ. The following well-known fact will be very useful.

Proposition 0.2. (i) *Let $\varkappa$ be an uncountable regular cardinal, and suppose S is stationary in $\varkappa$. If $f: S \to \varkappa$ is an arbitrary mapping, then there exists either stationary $S' \subseteq S$ and $\beta < \varkappa$, so that $f''(S') = \{\beta\}$, or there exists stationary $S'' \subseteq S$, such that $f \upharpoonright S''$ is strictly increasing.* (ii) *Let λ be an ordinal such that $cf(\lambda) > \omega$, and let $\alpha < cf(\lambda)$. If $S = \bigcup\{S_\beta \mid \beta < \alpha\}$ is stationary in λ, then some S_β is stationary in λ.*

The following well-known fact will be also useful.

Proposition 0.3. *If $\varkappa$ is an infinite cardinal, then there is a family X_α, $\alpha < 2^\varkappa$ of subsets of $\varkappa$, such that $\bigcup\{X_\alpha \mid \alpha \in U\} - \bigcup\{X_\alpha \mid \alpha \in V\}$ has cardinality $\varkappa$, for every disjoint non-empty finite U, $V \subseteq 2^\varkappa$.*

If I is a linear ordering and for every $i \in I$, L_i is a linear ordering, let $\Sigma\{L_i \mid i \in I\}$ denote the sum of the L_i's over I.

1. Some technical lemmas

Let $On(\omega) = \{\alpha \in On \mid \lim \alpha \,\&\, cf(\alpha) = \omega\}$, i. e. $On(\omega)$ is the class of all ordinals cofinal with ω. In advance, let for every $\alpha \in On(\omega)$ fix a continuous strictly increasing mapping $f_\alpha: \omega + 1 \to On$, such that $f_\alpha(\omega) = \alpha$.

For any nonempty set $S \subseteq On(\omega)$ let $L(S)$ denote the set $\{f_\alpha \mid \alpha \in S\}$, wich we always consider to be ordered lexicographically: $f \prec g$ iff $f(n) < g(n)$, where $n = \min\{m \mid f(m) \neq g(m)\}$. Let us agree till § 5 when working with any set of ordinals S we mean that $S \subseteq On(\omega)$ and $S \neq \varnothing$.

Lemma 1.1. *Let $\varkappa$ be an uncountable regular cardinal, and let S be a subset of $\varkappa$. Let $S' = \{\alpha \in S \mid$ there exists $n_\alpha < \omega$ such that $\{\beta \in S \mid f_\alpha \prec f_\beta$ and $f_\alpha \upharpoonright n \subset f_\beta\}$ is non-stationary$\}$. Then S' is non-stationary in $\varkappa$.*

Proof: Assume the contrary, i.e. that S' is stationary. In virtue of Prop. 0.2. we can assume $n_\alpha = n$, for every $\alpha \in S'$. But then by iterating Prop. 0.2. we obtain a stationary $S'' \subseteq S$, such that $f_\alpha \upharpoonright n = f_\beta \upharpoonright n$, for every α, $\beta \in S''$.

Since $L(S'')$ has no uncountable anti-well $\prec-$ ordered subset and since S'' is not the union of countably many non-stationary sets, there is $\alpha \in S''$, such that $\{\beta \in S'' \mid f_\alpha \prec f_\beta\}$ is stationary in $\varkappa$. But this contradicts the definition of S'.

Corollary 1.2. *Let $\varkappa$ be an uncountable regular cardinal, and suppose S is stationary in $\varkappa$. Then $L(S)$ is not the union of fewer than $\varkappa$ well ordered subsets.*

Lemma 1.3. *Let $\varkappa$ be an uncountable regular cardinal, and let $S, S' \subseteq \varkappa$. If $L(S)$ is order-isomorphic to a subset of $L(S')$, then $S - S'$ is non-stationary in $\varkappa$.*

Proof: Assume the contrary, i. e. that $S_0 = S - S'$ is stationary in $\varkappa$. Let $H: L(S) \to L(S')$ be a given isomorphism. The mapping $h: S_0 \to \varkappa$ is defined by $H(f_\alpha) = f_{h(\alpha)}$. Since h is one-to-one we infer by Prop. 0.2. that there is a stationary $S_1 \subseteq S_0$ such that $h \restriction S_1$ is strictly increasing. The set $h''(S_1)$ is non-stationary because $h^{-1} \restriction h''(S_1)$ is one-to-one and regressive. Therefore there is a club C such that $C \cap h''(S_1) = \varnothing$. Let $(c_\beta \mid \beta < \varkappa)$ be the normal enumeration of C (suppose $c_0 = 0$). To any $\alpha \in S_1$ corresponds $\beta(\alpha)$ such that $c_{\beta(\alpha)} < h(\alpha) < c_{\beta(\alpha)+1}$. By the Prop. 0.2. there is a stationary $S_2 \subseteq S_1$, such that $\beta(\alpha) \neq \beta(\alpha')$, for $\alpha, \alpha' \in S_2$, $\alpha \neq \alpha'$. Let $\alpha \in S_2$ be arbitrary and let us put

$$n(\alpha) = \min\{n \mid H(f_\alpha)(m) > c_{\beta(\alpha)}\}.$$

There exist a stationary $S_3 \subseteq S_2$ and $n < \omega$, such that $n_\alpha = n$, for every $\alpha \in S_3$. It is easy to see that $F(f_\alpha) = H(f_\alpha) \restriction n+1$ is an isomorphism from $L(S_3)$ into $({}^n\varkappa, \prec)$, what contradicts Cor. 1.2, because $({}^n\varkappa, \prec)$ is well ordered.

Lemma 1.4. *Let S be a set of ordinals (from $On(\omega)$) such that $S \cap \lambda$ is non-stationary in λ for every λ. Then $L(S)$ is the union of countably many well-ordered subsets.*

Proof: By induction on λ we shall prove that $L(S \cap \lambda)$ is the union of countably many well-ordered subsets, for every λ. Let $\lambda \in On$ and let $L(S \cap \lambda')$ be the union of $\leqslant \aleph_0$ well-ordered subsets, for every $\lambda' < \lambda$. Obviously, we may suppose that $cf(\lambda) > \omega$. Let then C be club in λ, so that $C \cap S = \varnothing$. Let $(c_\beta \mid \beta < cf(\lambda))$ be the normal enumeration of C (suppose $c_0 = 0$). By induction hypothesis, for each $\delta < \lambda$ there is a mapping $F_\delta: L(S \cap \delta) \to \omega$ such that for all n, $\{f \in L(S \cap \delta) \mid F_\delta(f) = n\}$ is well-ordered by $\prec$. For every $\alpha \in S$ there exists a unique $\beta(\alpha)$ such that $c_{\beta(\alpha)} < \alpha < c_{\beta(\alpha)+1}$. Now we define a mapping G which associates to each $f_\alpha \in L(S \cap \lambda)$ a pair $(n, m) \in \omega \times \omega$, where

$$n = \min\{k \mid f_\alpha(k) > c_{\beta(\alpha)}\} \quad \text{and} \quad m = Fc_{\beta(\alpha)+1}(f_\alpha).$$

It is easy to see that $\{f \in L(S \cap \lambda) \mid G(f) = (n, m)\}$ is well-ordered by $\prec$, for every $(n, m) \in \omega \times \omega$.

Remark. Observe that the Lemmas 1.1 — 1.4 do not depend on how at the begining we associate to every $\alpha \in On(\omega)$ the mapping $f_\alpha: \omega+1 \to On$.

If $\varkappa$ is an uncountable regular cardinal, and if S is stationary in $\varkappa$, then by lemma 1.1 there is $S' \subseteq S$, such that $S - S'$ is non-stationary and that $L(S')$ has the property that $\{\alpha \in S' \mid f \prec f_\alpha \prec g\}$ is stationary in $\varkappa$ for every $f, g \in L(S')$, $f \prec g$. This fact shall be expressed with the wording that in $L(S')$ *every interval is stationary.*

If $S \subseteq On(\omega)$ is stationary in some λ, then $tp(L(S), \prec)$ belongs to a class of ordered types that was studied in [*B*].

2. Very strongly rigid BA's exist in every regular cardinal $>\aleph_0$

Let $S \subseteq On(\omega)$, $S \neq \varnothing$. Then by $B(S)$ we denote the BA of all finite unions of intervals from $L(S)$ of the form $[x, y)$, $x, y \in L(S) \cup \{\pm\infty\}$, i.e. $B(S)$ is the interval algebra on $L(S)$. So, $|B(S)| = |S|$ for infinite S.

Lemma 2.1. *Let $\varkappa$ be an uncountable regular cardinal and let $S, S' \subseteq \varkappa$. If there exists a strictly increasing mapping H from $B(S)$ into $B(S')$, then $S - S'$ in non-stationary in $\varkappa$.*

Proof: Assume the contrary, i.e. that $S_0 = S - S'$ is stationary in $\varkappa$. For every $\alpha \in S_0$ we put $b_\alpha = (\cdot, f_\alpha)(\in B(S))$. So, $H(b_\alpha) \subset H(b_\beta)$, for every $\alpha, \beta \in S_0$, $f_\alpha \prec f_\beta$. Let $\alpha \in S_0$. Since $H(b_\alpha) \in B(S')$, there exists unique decomposition

$$H(b_\alpha) = \cup\{[x_\alpha^i, y_\alpha^i) \mid i < n(\alpha)\}, \tag{1}$$

where $n(\alpha) < \omega$, x_α^i, $y_\alpha^i \in L(S') \cup \{\pm\infty\}$, for every $i < n(\alpha)$ and $x_\alpha^i \prec y_\alpha^i \prec \prec x_\alpha^{i+1} \prec y_\alpha^{i+1}$, for every $i < n(\alpha) - 1$. Since S_0 is stationary, let us assume that for some $n < \omega$, $n(\alpha) = n$, for every $\alpha \in S_0$.

Let us define $h: S_0 \to \varkappa$ in the following way. Let $\alpha \in S_0$, then there exists a $\beta \in S'$, such that $x_\alpha^0 = f_\beta$ or $x_\alpha^0 = -\infty$. In the first case let us set $h(\alpha) = \beta$ and in the second case $h(\alpha) = 0$.

In virtue of Prop. 0.2 we know that either there is a stationary $S_1 \subseteq S_0$ and $\beta_1 < \varkappa$, satisfying $h''(S_1) = \{\beta_1\}$ or there exists a stationary $S_1' \subseteq S_0$ such that $h \restriction S_1'$ is one-to one. We claim that the second case does not hold.

Suppose, on the contrary that such an S_1' exists. Let $\alpha, \beta \in S_1'$ and $f_\alpha \prec f_\beta$. Since $H(b_\alpha) \subset H(b_\beta)$, we infer from (1) that $x_\alpha^0 \succcurlyeq x_\beta^0$. Since $h(\alpha) \neq h(\beta)$, we have $x_\alpha^0 = f_{h(\alpha)} \neq f_{h(\beta)} = x_\beta^0$ (suppose $0 \notin h''(S_1')$). Thus necessarily $f_{h(\alpha)} \succ f_{h(\beta)}$. This proves that $L(S_1')$ is anti-isomorphic to the subset of $L(S')$. But this is not possible because $L(S_1')$ contains an uncountable well-ordered set, while $L(S')$ does not contain uncountable anti-well-ordered subset.

Thus, there exist a stationary $S_1 \subseteq S_0$ and $\beta_1 < \varkappa$, so that $h''(S_1) = \{\beta_1\}$, i.e. $x_\alpha^0 = x_\beta^0$, for every $\alpha, \beta \in S_1$.

Let us now define $l: S_1 \to \varkappa$ in this way. Let $\alpha \in S_1$, then there is a $\beta \in S'$, such that $y_\alpha^0 = f_\beta$ or $y_\alpha^0 = +\infty$. In the first case let us put $l(\alpha) = \beta$ and $l(\alpha) = 0$ in the second case.

By Prop 0.2. we know that either there is a stationary $S_2 \subseteq S_1$ and $\beta_2 < \varkappa$, such that $l''(S_2) = \{\beta_2\}$ or there is a stationary $S_2' \subseteq S_1$, such that $l \restriction S_2'$ is strictly increasing. We claim that the second case does not hold.

Suppose, on the contrary, that such an S_2' exists. Let $\alpha, \beta \in S_2'$ and $f_\alpha \prec f_\beta$. Since $H(b_\alpha) \subset H(b_\beta)$ and $S_2' \subseteq S_1$, we infer from (1) that $y_\alpha^0 \preccurlyeq y_\beta^0$. Since $l(\alpha) \neq l(\beta)$, we have $y^0 = f_{l(\alpha)} \neq f_{l(\beta)} = y_\beta^0$ hence $f_{l(\alpha)} \prec f_{l(\beta)}$. This proves that $L(S_2')$ is isomorphic to a subset of $L(S')$, what is imposible by Lemma 1.3.

Thus there exists an $S_2 \subseteq S_1$ and $\beta_2 < \varkappa$, such that $l''(S_2) = \{\beta_2\}$, i.e. $y_\alpha^0 = y_\beta^0$, for every $\alpha, \beta \in S_2$.

Repeating this procedure $2n$ times we get a stationary set $S_{2n} \subseteq S_{2n-1} \subseteq \subseteq \cdots \subseteq S_1 \subseteq S$, such that $x_\alpha^i = x_\beta^i$ and $y_\alpha^i = y_\beta^i$, for every $i < n$ and every $\alpha, \beta \in S_{2n}$. This means that $H(b_\alpha) = H(b_\beta)$, for every $\alpha, \beta \in S_{2n}$ in contradiction with the assumption that H is strictly increasing. This finishes the proof.

L e m m a 2.2. *Let* $\varkappa$ *be an uncountable regular cardinal and let* $S, S' \subseteq \varkappa$. *If there exists a homomorphism* H *from* $B(S)$ *onto* $B(S')$, *then* $S'-S$ *is non-stationary in* $\varkappa$.

P r o o f: Since $B(S)$ is interval algebra it is by Theorem 1.5 from [Ro] retractive, what means that there exists a one-to-one homomorphism $G: B(S') \to B(S)$ so that $HG = id_{B(S')}$. By Lemma 2.1 we infer that $S'-S$ is non-stationary in $\varkappa$.

L e m m a 2.3. *Let* $\varkappa$ *be an uncountable regular cordinal and let* $S \subseteq \varkappa$ *be a stationary in* $\varkappa$, *such that every interval of* $L(S)$ *be stationary. Then there is no non-trivial strictly increasing mapping from* $B(S)$ *into* $B(S)$.

P r o o f: Assume the contrary, i.e. that there exists a non-trivial strictly increasing mapping $H: B(S) \to B(S)$. Thus, there exists a $b \in B(S)$, such that $c = H(b) \neq b$. Two cases are to be considered: $b - c \neq \varnothing$ and $b \subset c$. Consider the first case $b - c \neq \varnothing$. Let $S' = \{\gamma \in S \mid f_\gamma \in b - c\}$ and $S'' = \{\gamma \in S \mid f_\gamma \in c\}$. By hypothesis S' and S'' are disjoint stationary sets in $\varkappa$, $B(S') \cong B(S) \restriction b - c$ and $B(S'') \cong B(S) \restriction c$ in contradiction to the Lemma 2.1, because $H \restriction (B(S) \restriction b - c)$ is a strictly increasing mapping from $B(S) \restriction b - c$ into $B(S) \restriction c$. Analogously one proves that the second case does not occur.

Following theorem follows directly from Prop. 0.1, Lemma 1.1 and 2.3.

T h e o r e m 2.4. *Let* $\varkappa$ *be an uncountable regular cardinal, and suppose* S *is stationary in* $\varkappa$. *Then there exists* $S' \subseteq S$ *such that* $S - S'$ *is non-stationary and such that* $B(S')$ *is very strongly rigid BA of power* $\varkappa$.

Let $\varkappa > \aleph_0$ be a regular cardinal. Let R_α, $\alpha < \varkappa$ be a partition of $\{\alpha < \varkappa \mid cf(\alpha) = \omega\}$ into stationary subsets. By Prop. 0.3. there exists a family X_α, $\alpha < 2^\varkappa$ of subsets of $\varkappa$ no one of which is included in any other. Let $S_\alpha = \bigcup \{R_\beta \mid \beta \in X_\alpha\}$, for $\alpha < 2^\varkappa$. The family S_α, $\alpha < 2^\varkappa$ has the property that $S_\alpha - S_\beta$ is stationary for every α, $\beta < 2^\varkappa$, $\alpha \neq \beta$. Without depraving this property we can assume, omitting non-stationary subset of S_α, that every interval in $L(S_\alpha)$ is stationary, for every $\alpha < 2^\varkappa$. According to Lemmas 2.1 — 2.3 and Theorem 2.4 we have the following.

T h e o r e m 2.5. *Let* $\varkappa$ *be an uncountable regular cardinal. Then the family* $B(S_\alpha)$, $\alpha < 2^\varkappa$ *of BA's we have just constructed has the following properties:*

(i) $B(S_\alpha)$ *is very strongly rigid BA of power* $\varkappa$, *for every* $\alpha < 2^\varkappa$.

(ii) *If* $H: B(S_\alpha) \to B(S_\beta)$ *is either strictly increasing or homomorphism onto, then* $\alpha = \beta$ *and* $H = id_{B(S_\alpha)}$.

2. Some more properties of BA's of the form $B(S)$

(1) Let $\varkappa$ be an uncountable regular cardinal and let I be a well ordered set. Let S and S_i, $i \in I$ be non-empty subsets of $\varkappa$. Let $L = \Sigma\{L(S_i) \mid i \in I\}$ be the sum of lineary ordered sets $L(S_i)$ over I and let $B(L)$ be the interval algebra on L. The proof of Lemma 2.1 is easy transferrable to prove the following.

L e m m a 3.1. *Let* $\varkappa$, S *and* S_i, $i \in I$ *be as above. If there exists a strictly increasing or strictly decreasing mapping from* $B(S)$ *into* $B(L)$, *then* $S - \bigcup \bigcup \{S_i \mid i \in I\}$ is non-stationary in $\varkappa$.

(2) Let $S \subseteq On(\omega)$ be any set. Then there is λ such that $S \cap \lambda$ is stationary in λ iff there is $S' \subseteq S$, such that $B(S')$ is very strongly rigid BA.

Let $S \cap \lambda$ be stationary in some λ. Obviously $\varkappa = cf(\lambda) > \omega$; thus the existence of $S' \subseteq S \cap \lambda$ for which $B(S')$ is very stronly rigid follows from Theorem 2.4. Suppose now that $S \cap \lambda$ is non-stationary in λ for every λ. In virtue of Lemma 1.4 we know that $L(S)$ is the union of countably many well-ordered subsets. Let $S' \subseteq S$ be arbitrary. It is clear that we may assume that S' be uncountable. In virtue of the main result in [L] we know that there exists a sequencee (x_n, y_n), $n < \omega$ of nonempty disjoint intervals from $L(S')$ such that there is a one-to-one order-preserving function from (x_n, y_n) into (x_{n+1}, y_{n+1}), for every $o < \omega$. W. l. o. g. we may assume $y_n \leqslant x_{n+1}$, for every $n < \omega$. Let $a_n = [x_n, y_n)$ $n < \omega$ and let $b = L(S') - \{a_n \mid n < \omega\}$. Let $B_n = B(S') \upharpoonright a_n \cong$ interval algebra on a_n, $n < \omega$ and let $B = B(b) =$ interval algebra on b. Let $A = \{p \in B \times \times \Pi \{B_n \mid l \leqslant n < \omega\} \mid \text{support of } p \text{ is finite}\}$. Then it is easy to find F, G: $: B(S') \to A$, F is one-to-one and G is onto homomorphism such that $F \neq G$. So, $B(S')$ is not very strongly rigid BA.

(3) Let $\varkappa > \aleph_0$ be a regular cardinal, and let S be stationary in $\varkappa$ such that every interval in $L(S)$ be stationary. According to Lemma 2.3 we infer that $B(S)$ is very strongly rigid BA. For every $a \in B(S)$ we put $S_\alpha = \{\alpha \mid f_\alpha \in a\}$. Then $H(a) = [S_\alpha]$ is embedding of the BA $B(S)$ into the BA $D(\varkappa) \upharpoonright [S]$, where $[S] = \{S' \subseteq \varkappa \mid S \triangle S' \in I(\varkappa)\}$.

On the other side, from Lemmas 2.1 and 2.2 we inter that the algebras of the from $B(S)$, $S \subseteq \{\alpha < \varkappa \mid cf(\varkappa) = \omega\}$ with respect to the relations „is embeddable in" and „is homomorphic image of" behave like corresponding members of the algebra $D(\varkappa)$ wrt its ordering.

(4) Let $\varkappa > \aleph_0$ be a regular cardinal, and let S be stationary in $\varkappa$ such that every interval in $L(S)$ be stationary. Let $\overline{L}(S)$ be the Stone space of the BA $B(S)$. Let $\exp \overline{L}(S)$ be the set of all closed subsets of $\overline{L}(S)$ in Vietoris topology and let $\exp B(S)$ be the BA of all clopen subsets of $\exp \overline{L}(S)$. Then $\exp B(S)$ is BA of power $\varkappa$. Using the ideas of the proof of Lemma 2.1 we can prove that $\exp B(S)$ is mono-rigid BA. On the other side, using main theorem from [Tr] we may prove that $\exp B(S)$ is not embeddable in an interval algebra.

4. Very strongly rigid BA's in singular cardinalities

In § 2 we showed that very strongly rigid BA's exist in every regular cardinal $> \aleph_0$. In the present section we shall consider the problem of the existence of such algebras of singular power.

T h e o r e m 4.1. *Let* $\varkappa$ *be a singular cardinal and let* $2^\lambda \geqslant \varkappa$, *for some* $\lambda < \varkappa$. *Then there exists a family* B_α, $\alpha < 2^\varkappa$ *of BA's such that:*

(i) B_α *is very strongly rigid BA of power* $\varkappa$, *for every* $\alpha < 2^\varkappa$.

(ii) *If* $H: B_\alpha \to B_\beta$ *is either strictly increasing or homomorphism onto, then*

$$\alpha = \beta \quad \text{and} \quad H = id_{B_\alpha}.$$

Proof: Let $\lambda > \omega$ be a regular cardinal such that $2^\lambda \geqslant \varkappa$. Let R_α, $\alpha < \lambda$ be a family of pairwise disjoint stationary subsets of $\{\delta < \lambda \mid cf(\delta) = \omega\}$. At the beginning of § 2 we associate to every $\delta < \lambda$, $cf(\delta) = \omega$ a strictly increasing continuous mapping $f_\delta: \omega + 1 \to \lambda$, $f_\delta(\omega) = \delta$. Assume (only in this proof) that $\{\delta \in R_\alpha \mid f_\delta \supset g\}$ be stationary in λ, for every $\alpha < \lambda$ and every $g \in {}^{\underline{\omega}}\lambda$.

Let X_α, $\alpha < 2^\lambda$ be a family of subsets of λ such that $X_\alpha - \bigcup\{X_\beta \mid \beta \in U\} \neq \varnothing$, for every $\alpha < 2^\lambda$ and every finite set $U \subseteq 2^\lambda$, $U \not\ni \alpha$ (see Prop. 0.3). Put.

$S_\alpha = \bigcup\{R_\beta \mid \beta \in X_\alpha\}$, for every $\alpha < 2^\lambda$.

Let $Z \subseteq \varkappa$ be an arbitrary set of power $\varkappa$ and let $L_Z = \Sigma\{L(S_\alpha) \mid \alpha \in Z\}$ be the sum of linearly ordered sets $L(S_\alpha)$ over Z (with well-ordering induced from $\varkappa$). Let $B(L_Z)$ be the interval algebra determined by L_Z.

Let I_Z be the ideal in $B(L_Z)$ which is generated by the set $\{[x, y) \mid$ there is $\alpha \in Z$ such that $x, y \in L(S_\alpha)\}$. Put $B_Z = I_Z \cup -I_Z$. Then it is obvious that B_Z is a subalgebra of $B(L_Z)$ of power $\varkappa$.

Let K be a family of power $2^\varkappa$ of subsets of $\varkappa$ of power $\varkappa$ no one which is included in any other. Let us prove that the family B_Z, $Z \in K$ satisfies the conclusion of the theorem. According to Prop. 0.1 it is sufficient to prove the following:

If $H: B_Y \to B_Z$ is stricly increasing, then $Y = Z$ and $H = id_{B_Y}$.

Let $Y, Z \in K$ and let $H: B_Y \to B_Z$ is an one-to-one homomorphism. Let us prove $Y = Z$ (the proof of $H = id_{B_Y}$ is similar; therefore we omit it).

Assume the contrary, i.e. that some $\alpha \in Y - Z$ exists. Let $a \in I_Y$ and $a \subseteq L(S_\alpha)$ be non-empty. Suppose $H(a) \in I_Z$. This means that a finite set $U \subseteq Z$ exists, such that $H(a) \subseteq \bigcup\{L(S_\beta) \mid \beta \in U\}$. Let $S_a = \{\delta < \lambda \mid f_\delta \in a\}$. Then in basis of the above construction we know that $S_a - \bigcup\{S_\beta \mid \beta \in U\}$ is stationary in λ. But $H \restriction (B_Y \restriction a)$ could be meant as strictly increasing mapping from $B(S_a)$ into $B(\Sigma\{L(S_\beta) \mid \beta \in U\})$ what contradicts Lemma 3.1.

Let us suppose now that $H(a) \in -I_Z$, for every non-empty $a \subseteq L(S_\alpha)$, $a \in I_Y$. Let $\varnothing \neq a \subseteq L(S_\alpha)$, $a \in I_Y$ be such that $S_b = \{\delta \in S_\alpha \mid f_\delta \notin a\}$ is stationary in λ, i. e. such that $b = L(S_\alpha) - a$ contains a nonempty interval. Then $-H(a) \in I_Z$ what means that there is a finite $V \subseteq Z$ such that $-H(a) \subseteq \bigcup\{L(S_\beta) \mid \beta \in V\}$. In virtue of the above conclusions we infer that $S_b - \bigcup\{S_\beta \mid \beta \in V\}$ is stationary in λ. The mapping $F(c) = -H(a \cup c)$, $c \in B_Y$, $c \subseteq b$ could be considered as strictly decreasing mapping from $B(S_b)$ into $B(\Sigma\{L(S_\beta) \mid \beta \in V\})$, in contradiction with Lemma 3.1. This finishes the proof.

Let now $\varkappa$ be a fixed singular cardinal and let λ_α, $\alpha < cf(\varkappa)$ be a fixed strictly increasing sequence of cardinals with supremum $\varkappa$, such that $\lambda_\alpha > cf(\varkappa)$ for every $\alpha < cf(\varkappa)$.

Lemma 4.2. *Let* $S, S' \subseteq \varkappa$ *be such that* $S \cap \beta$ *and* $S' \cap \beta$ *be non-stationary in* β, *for every* $\beta \notin \{\lambda_\alpha^+ \mid \alpha < cf(\varkappa)\}$. *If there exists a strictly increasing mapping* $H: B(S) \to B(S')$ *then* $(S - S') \cap \lambda_\alpha^+$ *is non-stationary in* λ_α^+, *for every* $\alpha < cf(\varkappa)$.

Proof: Assume the contrary, i. e. that for some $\alpha < cf(\varkappa)$ the set $S_0 = (S - S') \cap \lambda_\alpha^+$ be stationary in λ_α^+. For every $\delta \in S_0$ we put $b_\delta = (\cdot, f_\delta)$

($\in B(S)$). So, $H(b_\delta)\subset H(b_\gamma)$, for every $\delta, \gamma\in S_0$, $f_\delta\prec f_\gamma$. Let $\delta\in S_0$. Since $H(b_\delta)\in B(S')$, there exists a unique decomposition

$$(2) \qquad H(b_\delta)=\cup\{[x_\delta^i, y_\delta^i)\mid i<n(\delta)\},$$

where $n(\delta)<\omega$, $x_\delta^i, y_\delta^i\in L(S')\cup\{\pm\infty\}$, for every $i<n(\delta)$ and $x_\delta^i\prec y_\delta^i\prec x_\delta^{i+1}\prec y_\delta^{i+1}$, for every $i<n(\delta)-1$. Since S_0 is stationary, let us assume that for some $n<\omega$, $n(\delta)=n$, for every $\delta\in S_0$.

Let us define $h: S_0\to\varkappa$ in the following way. Let $\delta\in S_0$. Then either there exists a $\gamma\in S'$, such that $x_\delta^0=f_\gamma$ or $x_\gamma^0=-\infty$. In the first case let us put $h(\delta)=\gamma$ and in the second $h(\delta)=0$.

In virtue of Prop. 0.2. (really a small extension of it) we know that either there exists a stationary $S_1\subseteq S_0$ and $\gamma_1<\varkappa$, such that $h''(S_1)=\{\gamma_1\}$ or there exists a stationary $S_1'\subseteq S_0$ such that $h\restriction S_1'$ is one-to-one. As in the proof of Lemma 2.1 one proves that the second case is not possible.

Consequently, there exists a stationary $S_1\subseteq S_0$, such that $x_\delta^0=x_\gamma^0$, for every δ, $\gamma\in S$.

Let us now define $l: S_1\to\varkappa$. Let $\delta\in S_1$. Then either there is a $\gamma\in S'$ such that $y_\delta^0=f_\gamma$ or $y_\delta^0=+\infty$. Let $l(\delta)=\gamma$ in the first case and $l(\delta)=0$ in the second case. As above we know that either there exists a stationary $S_2\subseteq S_1$ and $\gamma_2<\varkappa$, a such that $l''(S_2)=\{\gamma_2\}$ or there exists a stationary (in λ_δ^+) $S_2'\subseteq S_1$, such that $l\restriction S_2'$ is one-to-one. We claim that the last case is not possible.

Suppose the contrary, i.e. that such an S_2' exists. Let δ, $\gamma\in S_2'$ and $f_\delta\prec f_\gamma$. Since $H(b_\delta)\subset H(b_\gamma)$ and $S_2'\subseteq S_1$, we infer from (2) that $y_\delta^0\prec y_\gamma^0$. Since $l(\delta)\neq l(\gamma)$ one has $y_\delta^0=f_{l(\delta)}\neq f_{l(\gamma)}=y_\gamma^0$ thus $f_{l(\delta)}\prec f_{l(\gamma)}$, what proves that $L(S_2')$ is similar ta a subset of $L(S')$. By Lemma 1.3 $\{\delta\in S_2'\mid l(\delta)<\lambda_\alpha^+\}$ is non-stationary in λ_α^+. But even the set $\{\delta\in S_2'\mid l(\delta)>\delta_\alpha^+\}$ cannot be stationary in λ_α^+, because $\{f_\delta\mid \delta\in S_2' \,\&\, l(\delta)>\lambda_\alpha^+\}$ is similar to $\{f_{l(\delta)}\mid \delta\in S_2' \,\&\, l(\delta)>\lambda_\alpha^+\}$ and because $l''(S_2')\cap\beta$ is non-stationary in β for every $\beta>\lambda_\alpha^+$ (see Cor. 1.2 and Lemma 1.4).

Consequently there exists a stationary $S_2\subseteq S_1$, such that $y_\delta^0=y_\gamma^0$, for every δ, $\gamma\in S_2$.

Repeating this procedure $2n$ times we get a stationary (in λ_α^+) $S_{2n}\subseteq\subseteq S_{2n-1}\subseteq\cdots\subseteq S_1\subseteq S_0$, such that $x_\delta^i=x_\gamma^i$ and $y_\delta^i=y_\gamma^i$, for every $i<n$ and δ, $\gamma\in S_{2n}$. This means that $H(b_\delta)=H(b_\gamma)$, for every δ, $\gamma\in S_{2n}$ what contradicts the assumption that H is strictly increasing.

Next lemmas follow from the Lemma 4.2 in a similar way as we deduced Lemmas 2.2. and 2.3. from Lemma 2.1.

Lemma 4.3. *Let $\varkappa$, λ_α, $\alpha<cf(\varkappa)$ and S, $S'\subseteq\varkappa$ be as in Lemma 4.2. If there exists homomorphism H from $B(S)$ onto $B(S')$, then $(S'-S)\cap\lambda_\alpha^+$ is non-stationary in λ_α^+, for every $\alpha<cf(\varkappa)$.*

Lemma 4.4. *Let $\varkappa$ and λ_α, $\alpha<cf(\varkappa)$ as above. Let $S\subseteq\varkappa$ has the property that for every non-empty $a\in B(S)$ there exists an $\alpha<cf(\varkappa)$, such that $\{\delta\in\in S\mid f_\delta\in a\}\cap\lambda_\alpha^+$ is stationary in λ_α^+. Then there is no non-trivial strictly increasing mapping from $B(S)$ into $B(S)$.*

Let $\lambda>\aleph_0$ be a regular cardinal. Let $E(\lambda)$ denote the following statement: There is an $S\subseteq\{\lambda<\delta\mid cf(\delta)=\omega\}$ stationary in λ such that $S\cap\alpha$ is not stationary in α, for every $\alpha<\lambda$.

Since $\square_\lambda$ implies $E(\lambda^+)$ (see, e. g. [KM]), in virtue of [J] p. 286, we infer that $E(\lambda^+)$ holds if λ^+ is not Mahlo in L.

Theorem 4.5. *Assume $E(\lambda^+)$ for every λ. Let $\varkappa$ be an arbitrary cardinal $>\aleph_0$. Then there exists a family B_α, $\alpha<2^\varkappa$ of interval BA's such that:*

(i) *B_α is very strongly rigid BA of power $\varkappa$, for every $\alpha<2^\varkappa$.*

(ii) *If $H: B_\alpha \to \beta_\beta$ is either strictly increasing or homomorphism onto, then*

$$\alpha=\beta \quad \text{and} \quad H=id_{B_\alpha}.$$

Proof: Let $\varkappa>\aleph_0$. In virtue of Theorem 2.5 we may assume that $\varkappa$ is singular. Fix a strictly increasing sequence λ_α, $\alpha<cf(\varkappa)$ of cardinals with supremum $\varkappa$, such that $\lambda_\alpha>cf(\varkappa)$, for every $\alpha<cf(\varkappa)$.

By assumption, for every $\alpha<cf(\varkappa)$, we can find a sequence $S_{\alpha\beta}$, $\beta<2^{\lambda_\alpha^+}$ of stationary in λ_α^+ subset of $\{\delta \mid \lambda_\alpha<\delta<\lambda_\alpha^+ \,\&\, cf(\delta)=\omega\}$, such that $S_{\alpha\beta}\cap\gamma$ is non-stationary in γ, for every $\gamma<\lambda_\alpha^+$ and so that $S_{\alpha\beta}-S_{\alpha\beta'}$ is stationary in λ_α^+ for every β, $\beta'<2^{\lambda_\alpha^+}$, $\beta\neq\beta'$. Assume that $L(S_{\alpha\beta})$ has no end-points and that in $L(S_{\alpha\beta})$ every interval is stationary, for every $\alpha<cf(\varkappa)$ and $\beta<2^{\lambda_\alpha^+}$.

Let $p\in\Pi\{2^{\lambda_\alpha^+} \mid \alpha<cf(\varkappa)\}$. Then we set $S_p=\bigcup\{S_{\alpha p(\alpha)} \mid \alpha<cf(\varkappa)\}$.

Then $B(S_p)$, $p\in\Pi\{2^{\lambda_\alpha^+} \mid \alpha<cf(\varkappa)\}$ is a requested family of BA's as follows from Lemmas 4.2—4.4 and Prop. 0.1.

5. Ontorigid BA's exist in every uncountable cardinality

Let $\varkappa>\lambda$ be regular cardinals. For every $\delta<\varkappa$, $cf(\delta)=\lambda$ let us fix a strictly increasing continuous mapping $f_\delta: \lambda+1\to\varkappa$, such that $f_\delta(\lambda)=\delta$. To every non-empty set $S\subseteq\{\delta<\varkappa \mid cf(\delta)=\lambda\}$ we associate the set $L(S)=\{f_\delta \mid \delta\in S\}$ ordered lexicographically as well as the BA $B(S)$ of all finite union of intervals from $L(S)$ of the form $[x, y)$.

We shall always assume that the above mapping $\delta\to f_\delta$ has the following property:

$$(*)\qquad |\{f_\delta\restriction\alpha \mid \alpha\leqslant\lambda,\ f_\delta(\alpha)<\beta,\ \delta<\varkappa \text{ and } cf(\delta)=\lambda\}|<\varkappa,$$

for every $\beta<\varkappa$.

Proofs of next two lemmas are almost identical to the proofs of Lemmas 1.2 and 1.3, respectively.

Lemma 5.1. *Let $\varkappa>\lambda$ be regular cardinals and let $S\subseteq\{\delta<\varkappa \mid cf(\delta)=\omega\}$. Let $S'=\{\delta\in S \mid$ there exists $\alpha<\lambda$ such that $\{\gamma\in S \mid f_\delta<f_\gamma$ and $f_\delta\restriction\alpha\subset f_\gamma\}$ is non-stationary in $\varkappa\}$. Then S' is non-stationary in $\varkappa$.*

Lemma 5.2. *Let $\varkappa>\lambda$ be regular cardinals and let $S, S'\subseteq\{\delta<\varkappa \mid cf(\delta)=\lambda\}$. If $L(S)$ is order-isomorphic to a subset of $L(S')$, then $S-S'$ is non-stationary in $\varkappa$.*

Proofs of next two lemmas are almost identical to the proofs of Lemmas 2.1 and 2.3, respectively.

Lemma 5.3. *Let $\varkappa > \lambda$ be regular cardinals and let $S, S' \subseteq \{\delta < \varkappa \mid cf(\delta) = \lambda\}$. If there exists a strictly increasing mapping from $B(S)$ into $B(S')$, then $S - S'$ is non-stationary in $\varkappa$.*

Lemma 5.4. *Let $\varkappa > \lambda$ be regular cardinals and let $S \subseteq \{\delta < \varkappa \mid cf(\delta) = \lambda\}$ be a stationary in $\varkappa$, such that every interval of $L(S)$ be stationary. Then there is no non-trivial strictly increasing mapping from $B(S)$ into $B(S)$.*

Now we are ready for the proof of the main theorem of this section.

Theorem 5.5. *For every uncountable cardinal $\varkappa$, there exists a family B_α, $\alpha < 2^\varkappa$ such that:*

(i) *B_α is onto-rigid BA of power $\varkappa$, for every $\alpha < 2^\varkappa$.*

(ii) *If $H: B_\alpha \to \beta_\beta$ is onto homomorphism, then $\alpha = \beta$ and $H = id_{B_\alpha}$.*

Proof: Let $\varkappa > \aleph_0$. In virtue of the Theorems 2.5 and 4.1 we may assume that $\varkappa$ is a strongly limit singular cardinal. Therefore we can consider strictly increasing sequences $\varkappa_\alpha$, $\alpha < cf(\varkappa)$ and λ_α, $\alpha < cf(\varkappa)$ of regular cardinals with supremum $\varkappa$, so that $(2^{\lambda_\alpha})^+ = \varkappa_\alpha < \lambda_\beta$, for every $\alpha < \beta < cf(\varkappa)$.

Let us fix $\alpha < cf(\varkappa)$. Let $R_{\alpha\xi}$, $\xi < \varkappa_\alpha$ be a sequence of disjoint stationary in $\varkappa_\alpha$ subsets of $\{\delta < \varkappa_\alpha \mid cf(\delta) = \lambda_\alpha\}$. To every δ, $\varkappa_\alpha^- < \delta < \varkappa_\alpha$, $cf(\delta) = \lambda_\alpha$ we associate a strictly increasing continuous mapping $f_\delta: \lambda_\alpha + 1 \to \varkappa_\alpha$, such that $f_\delta(\lambda_\alpha) = \delta$. In virtue of the relation between λ_α and $\varkappa_\alpha$ it is obvious that we can assume that $\{\delta \in R_{\alpha\xi} \mid f_\delta \supset g\}$ is stationary in $\varkappa_\alpha$ for every strictly increasing and continuous $g: \beta \to \varkappa_\alpha$, $\beta < \lambda_\alpha$ and for every $\xi < \varkappa_\alpha$. Let us notify that the condition (*) is satisfied also.

Let $X_\alpha \subseteq \varkappa_\alpha$ and $X_\alpha \neq \varnothing$ and let $S_\alpha = \bigcup \{R_{\alpha\xi} \mid \xi \in X_\alpha\}$. The Stone space $\bar{L}(S_\alpha)$ of the BA $B(S_\alpha)$ is obtained from the Dedekind's completion of the linearly ordered set $L(S_\alpha)$ by doubling every nonend-point from $L(S_\alpha)$.

Let $x \in \bar{L}(S_\alpha)$; then the left $\chi^-(x, \bar{L}(S_\alpha))$ and the right $\chi^+(x, \bar{L}(S_\alpha))$ character of x in $\bar{L}(S_\alpha)$ are defined as usually. A simple consideration of the lexicographical ordering of $L(S_\alpha)$ shows that if some point $x \in \bar{L}(S_\alpha)$ has left (right) character $< \lambda_\alpha$ then its right (left) character necessarily equals either λ_α or $\varkappa_\alpha$. Also, one checks easily that the set of all points $x \in \bar{L}(S_\alpha)$ for which $0 < \chi^-(x, \bar{L}(S_\alpha)) < \chi^+(x, \bar{L}(S_\alpha))$ or $0 < \chi^+(x. \bar{L}(S_\alpha)) < \chi^-(x, \bar{L}(S_\alpha)$ is dense in $\bar{L}(S_\alpha)$.

Let $L = \Sigma\{L(S_\alpha) \mid \alpha < cf(\varkappa)\}$ be the sum of $L(S_\alpha)$'s over $cf(\varkappa)$ and $B(L)$ be the interval algebra on L. Then $B(L)$ is BA of power $\varkappa$; its Stone space $\bar{L}$ is obtained from the Dedekind's completion of L by doubling every nonend-point of L. We shall consider $L(S_\alpha)$ as a convex subset of $\bar{L}$, for every $\alpha < cf(\varkappa)$.

Let us prove that $B(L)$ is onto-rigid BA; the remainder of the theorem is proved as in other cases in this paper.

Assume the contrary, i.e. that there exists a nontrivial homomorphism H from $B(L)$ onto $B(L)$. Since by Lemma 5.4 $B(S_\alpha)$, $\alpha < cf(\varkappa)$ are onto-rigid BA's we infer easily that there are $\beta < \alpha < cf(\varkappa)$ and non void $a, b \in B(L)$, $a \subseteq \bar{L}(S_\alpha)$, $b \subseteq \bar{L}(S_\beta)$, such that $H(a) = b$ (there we identify $B(L)$ with BA of all clopen subsets of $\bar{L}$). Let $\bar{H}: \bar{L} \to \bar{L}$ be one-to-one continuous function which is dual to H. Let $x \in b$ and let $0 < \chi^-(x, \bar{L}) < \chi^+(x, \bar{L})$. Let $\bar{H}(x) = y (\in a)$. Since $\bar{H}$ is one-to-one and continuous one checks easily that $\{\chi^-(x, \bar{L}), \chi^+(x, \bar{L})\} = \{\chi^-(y, \bar{L}), \chi^+(y, \bar{L})\}$ hence $\chi^-(y, \bar{L}), \chi^+(y, \bar{L}) \leqslant \varkappa_\beta < \lambda_\alpha$. But this contradicts the quoted property of the Stone space $\bar{L}(S_\alpha)$. This finishes the proof.

Let A be a σ-complete BA. A is said to be σ-hyper-rigid (see [Bo 3]) whenever for every σ-complete algebra B, every σ-complete homomorphisms F and G from A into B, such that F is one-to-one and G is onto we have $A = G$.

Let $\varkappa > \aleph_1$ be a regular cardinal such that $\lambda^{\aleph_0} < \varkappa$, for every $\lambda < \varkappa$. For every $\delta < \varkappa$, $cf(\delta) = \omega_1$ we select again $f_\delta : \omega_1 + 1 \to \varkappa$ in the above way. Let $S \subseteq \{\delta < \varkappa \mid cf(\delta) = \omega_1\}$ be stationary set in $\varkappa$ such that every interval in $L(S)$ is stationary (observe that (*) holds). Let $B^\sigma(S)$ be the σ-completion of the BA $B(S)$ and let $L^\sigma(S)$ be the σ-completion of $L(S)$. The algebra $B^\sigma(S)$ has a nice representation as a subalgebra of the algebra of all regular open subsets of $L^\sigma(S)$ (see [Bo 3]). Using the analog of the Lemma 5.3 for algebras of the form $B^\sigma(S)$ we can prove thet $B^\sigma(S)$ is σ-hyper-rigid BA of power $\varkappa$.

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