

## SOME RESULTS IN THE FIXED POINT THEORY, II

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**Abstract.** The purpose of the present paper is to prove one theorem in which we have omitted the assumption of the completeness of the space from each. We have obtained the same conclusion as in Banach's theorem but with different sufficient conditions. In this paper fixed point theorems have been established for the mapping which are contractive over two consecutive elements of an orbit. The similar theorem is obtained for Banach spaces. Also, we prove a fixed point theorems of localization type.

### 1. Introduction and some results

In recent years a number of generalizations of the wellknown Banach contraction principle have appeared in the literature where the authors have introduced mappings of contractive type and studied the existence of their fixed points. A comparative study of these generalizations has, been made more recently by Rhoades [13] and Tasković [14].

Let  $T: X \rightarrow X$  be a mapping of a metric space  $(X, \rho)$  into itself. For  $x \in X$ , let us denote the subset  $\{x, Tx, \dots, T^k x\}$ ,  $k = 1, 2, \dots$ , of  $X$  by  $O(x, k)$  and the diameter of  $O(x, k)$  by  $\delta[O(x, k)]$ . For  $x, y \in X$  we put  $I(X, T) := \{x \in X \mid Tx = x\}$ ,  $\delta[O(x, \infty)] := \text{diam} \{x, Tx, T^2 x, \dots\}$ ,  $\delta[O(x, y, \infty)] := \text{diam} \{x, y, Tx, Ty, T^2 x, T^2 y, \dots\}$ . A space  $X$  is said to be  $T$ -orbitally complete iff every Cauchy sequence which is contained in  $O(x, \infty)$  for some  $x \in X$  converges in  $X$  (cf. [14]).

In [14] we introduced the concept of a generalized  $\phi$ -contraction  $T$  of a metric space  $X$  into itself i.e. of a mapping  $T: X \rightarrow X$  such that for all  $x, y \in X$

$$\rho[Tx, Ty] \leq \phi(\rho[x, y], \rho[x, Tx], \rho[y, Ty], \rho[y, Tx], \rho[x, Ty])$$

where the existing mapping  $\phi: (\mathbf{R}_+^0)^5 \rightarrow \mathbf{R}_+^0 := [0, +\infty)$  is increasing and has the property  $(\forall t \in \mathbf{R}_+ := (0, +\infty)) \limsup_{z \rightarrow t+0} \phi(z, \dots, z) < t$ .

On the other hand in [15] we introduced the concept of a diametral  $\phi$ -contraction  $T$  of a metric space  $X$  into itself i.e. of a mapping  $T: X \rightarrow X$  such that for every  $x, y \in X$ ,

$$(A) \quad \rho[Tx, Ty] \leq \phi(\delta[O(x, y, \infty)]), \quad \delta[O(x, \infty)] \in \mathbf{R}_+,$$

where the existing mapping  $\phi: \mathbf{R}_+^0 \rightarrow \mathbf{R}_+^0$  has the properties

$$(\forall t \in \mathbf{R}_+) (\phi(t) < t \wedge \limsup_{z \rightarrow t+0} \phi(z) < t).$$

It may be noted that generalized  $\varphi$ -contraction implies diametral  $\varphi$ -contractive mappings.

And finally, at the next step we prove a very general fixed point theorem which generalizes a great number of known results (see. [15]).

**Theorem 1.** *Let  $T$  be a diametral  $\varphi$ -contraction on a metric space  $X$  and let  $X$  be  $T$ -orbitally complete. Then for each  $x \in X$ , the sequence  $\{T^n x\}$  converges to a unique fixed point  $\xi$  of  $T$ . The velocity of this convergence is not necessarily geometrical.*

The proof of this theorem is based upon the fundamental lemma, proved in [14].

**Lemma 1.** *Let the mapping  $\varphi: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  have the properties  $(\forall t \in \mathbf{R}_+) \varphi(t) \leq t$  and  $\limsup_{z \rightarrow t+0} \varphi(z) < t$  for  $t \in \mathbf{R}_+$ . If the sequence  $(x_n)$  of nonnegative real numbers satisfy the condition  $x_{n+1} \leq \varphi(x_n)$ ,  $n = 1, 2, \dots$ ; then the sequence  $(x_n)$  tends to zero. The velocity of this convergence is not necessarily geometrical.*

*Proof of Lemma 1.* Since  $(x_n)$  is a nonincreasing sequence in  $\mathbf{R}_+$ , there is a  $t \geq 0$  such that  $x_n \rightarrow t$  ( $n \rightarrow \infty$ ). We claim that  $t = 0$ . If  $t > 0$ , then

$$t = \limsup_{n \rightarrow \infty} x_{n+1} \leq \limsup_{n \rightarrow \infty} \varphi(x_n) \leq \limsup_{z \rightarrow t+0} \varphi(z) < t,$$

which is a contradiction. Consequently  $t = 0$ , and  $\lim x_n = 0$ .

**Proof of Theorem 1.** For  $x_0 = x \in X$ , let  $x_n = T^n x$ , ( $n = 0, 1, 2, \dots$ ). It is easy to verify that the sequence  $\{x_n\}$  satisfies condition  $\delta[O(x_{n+1}, \infty)] \leq \varphi(\delta[O(x_n, \infty)])$ ,  $n = 0, 1, 2, \dots$  and hence applying Lemma 1. to the sequence  $\{\delta[O(x_n, \infty)]\}$  we obtain  $\lim \delta[O(x_n, \infty)] = 0$ . This implies that  $\{T^n x\}$  is a Cauchy sequence in  $X$ , and hence, by  $T$ -orbitally completeness, there is a  $\xi \in X$  such that  $x_n = T^n x \rightarrow \xi$  ( $n \rightarrow \infty$ ). Put  $x_n = T^n \xi$  ( $n = 0, 1, 2, \dots$ ). Since  $\{y_n\}$  is a bounded sequence of nonnegative reals, for some  $\varepsilon_0 \geq 0$ ,  $\delta[O(x_n, y_n, \infty)] \rightarrow \varepsilon_0$  ( $n \rightarrow \infty$ ). Similarly we have  $\varepsilon_0 = 0$ . Thus  $\xi = \lim y_n$  and by our Lemma 1. we have  $\delta[O(\xi, \infty)] = 0$  and it means that  $\xi$  is a fixed point of  $T$ . From (A) we have that  $\xi \in X$  is unique.

We are now able to prove the theorem in which the completeness of the space is replaced by its boundedness.

**Theorem 2.** *Let  $T$  be a diametral  $\varphi$ -contraction on a metric space  $X$  and let  $X$  be bounded. Then:*

(a) *The equality*

$$I(X, T) = T(A) = A := \bigcap_{n \in \mathbf{N}} T^n(X),$$

*holds;*

(b) *The set  $I(X, T)$  is either empty or contains exactly one element, i.e.  $T$  either has no fixed points or has exactly one fixed point.*

**Proof.** Since the inclusions  $I(X, T) \subset T(A) \subset A$  are true in the case of the mapping of the nonempty set  $X$  into itself, it suffices to prove the inverse inclusions. Let  $A \neq \emptyset$  and  $x, y \in A$ , then for each  $n \in \mathbb{N}$  there exist elements  $x_n, y_n \in X$  such that  $x = T^n(x_n)$ ,  $y = T^n(y_n)$  and

$$\text{diam } A \leq \rho[x, y] = \rho[T^n x_n, T^n y_n] \leq \varphi(\delta[O(T^{n-1} x_n, T^{n-1} y_n, \infty)]),$$

other we have

$$\text{diam } A \leq \delta[O(T^n x_n, T^n y_n, \infty)] \leq \varphi(\delta[O(T^{n-1} x_n, T^{n-1} y_n, \infty)]),$$

and applying the Lemma 1. to the sequence  $\{\delta[O(T^n x_n, T^n y_n, \infty)]\}$ , we obtain  $\delta[x, y] = 0$  when  $n \rightarrow \infty$ , i.e.  $\text{diam } A = 0$ . Consequently if  $A \neq \emptyset$  then  $A$  contains exactly one element, i.e.  $A = \{a\}$ . Since  $I(X, T) \subset T(A) \subset A$  we have  $T(a) \in \{a\}$  i.e.  $T(a) = a$ . Therefore  $A = \{a\} \subset A$  and this together with the above inclusions gives  $I(X, T) = T(A) = A = \{a\}$ . When  $A = \emptyset$  the equality  $I(X, T) = T(A) = A = \emptyset$  follows immediately from  $I(X, T) \subset T(A) \subset A$ .

**Remark.** In general we remark that the following relation hold  $I(X, T) \subset T(A) \subset A$ . None of the previous inclusions can be replaced by the equality, and the first one even not if it is assumed that  $T$  is a diametral  $\varphi$ -contraction of complete metric space  $X$ .

**Example 1.** Let  $X = \mathbb{R}$  be the set of real numbers  $\mathbb{R}$  with usual metric and assume  $T(x) = x/3$  ( $x \in X$ ). Then  $T$  is a diametral  $\varphi$ -contraction of complete metric space  $X$ . In that case  $I(X, T) = \{0\}$  and  $T(X) = X$ , thus  $T^n(X) = X$  ( $n \in \mathbb{N}$ ). This implies  $A = X$ , what leads eventually to  $T(A) = T(X) = X$ . This example shows that first inclusion cannot be replaced by equality if  $T$  is a diametral  $\varphi$ -contraction of a complete metric space  $X$ . Let

$$X = \bigcup_{n \in \mathbb{N}} \{a_v^{(n)} : 1 \leq v \leq n\} \cup \{a, b\},$$

and assume  $T: X \rightarrow X$ ,  $T(a_v^{(n)}) = T(a_{v+1}^{(n)})$ , ( $v = 1, \dots, n-1$ ;  $n = 2, 3, \dots$ ),  $T(a_n^{(n)}) = a$  ( $n = 1, 2, \dots$ )  $T(a) = T(b) = b$ . In that case  $A = \{a, b\}$  and  $T(A) = \{b\}$ .

**Remark.** This idea of theorem is due to D. Adamović.

## 2. Some localizations

In paper [15] we introduced the concept of a *locally  $\varphi$ -contraction  $T$  of a metric space  $X$  into itself* i.e. of a mapping  $T: X \rightarrow X$  such that for every  $x \in X$ ,

$$(B) \quad \rho[Tx, T^k x] \leq \varphi(\delta[O(x, k)]), \quad k = 2, 3, \dots$$

where the existing mapping  $\varphi: \mathbb{R}_+^0 \rightarrow \mathbb{R}_+^0$  has the properties

$$(\forall t \in \mathbb{R}_+) (\varphi(t) < t \wedge \limsup_{z \rightarrow t+0} \varphi(z) < t)$$

It may be noted that  $\varphi$ -contraction implies locally  $\varphi$ -contraction but not conversely.

Now, we can formulate a corresponding statement for locally  $\varphi$ -contractive mappings.

**Theorem 3.** (Tasković [15]) *Let  $T$  be a orbitally continuity locally  $\varphi$ -contraction on a metric space  $X$ . Then*

(a)  $\{T^n x\}$  is a Cauchy sequence for each  $x \in X$  with bounded  $T$ -orbit, that is, with  $\delta[O(x, \infty)] \in \mathbf{R}_+$ ,

(b) If set  $S := \{t \in \mathbf{R}_+ \mid t - \varphi(t) \leq \rho[x, Tx], x \in X\}$  is bounded, then  $\delta[O(x, \infty)] \in \mathbf{R}_+$ ,

(c) If  $x \in X$ ,  $\delta[O(x, \infty)] \in \mathbf{R}_+$  and the closure of  $O(x, \infty)$  is complete, then the sequence  $\{T^n x\}$  converges to a fixed point  $\xi$  of  $T$ .

*Some remarks*

1) Orbital and localization theorems of the class of fixed points contractions mapping doesn't say anything about the uniqueness of the fixed point. One example is sufficient to prove this.

**Example 2.** Let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be defined as follows  $T(x, y) = (x, \alpha y)$ ,  $\alpha \in (0, 1)$ . Then the point  $(x, 0)$  is a fixed point of the mapping  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ .

2) Note that one cannot delete a condition of orbital continuity of  $T$  in the Theorem 3., even  $T$  satisfied the stronger conditions of (B)

$$\rho[Tx, T^2 x] \leq \alpha \rho[x, Tx], x \in X,$$

for some  $\alpha \in [0, 1)$ . The following example shows it.

**Example 3.** Let  $X = \{0, 2^{-n}\}$ ,  $n = 1, 2, \dots$ ;  $T(2^{-n}) = 2^{-n-1}$ ,  $T(0) = 1$ . For  $x = 2^{-n}$  is  $\rho[Tx, T^2 x] = \rho[2^{-n-1}, 2^{-n-2}] = 2^{-n-2} = 2^{-1} \rho[2^{-n}, 2^{-n-1}] = 2^{-1} \rho[x, Tx]$ ; and for  $x = 0$   $\rho[Tx, T^2 x] = \rho[1, 2^{-1}] = 2^{-1} \rho[0, 1] = 2^{-1} \rho[x, Tx]$ . Then  $T$  satisfied stronger conditions with  $\alpha = 2^{-1}$ , but has not a fixed point

The assumption of continuity can be avoided if an additional assumption for the mapping is made. However, the unicity cannot be obtained.

**Theorem 4.** *Let  $T$  be a mapping of a metric space  $X$  into itself and let  $X$  be  $T$ -orbitally complete. Let for every  $x \in X$  and  $i, j \in \{0, 1, \dots, n\}$*

$$(C) \quad \rho[T^i x, T^j x] \leq \varphi(\delta[O(x, n)]), \quad n \in \mathbf{N}, \text{ and } \delta[O(x, \infty)] \in \mathbf{R}_+.$$

where the existing mapping  $\varphi: \mathbf{R}_+^0 \rightarrow \mathbf{R}_+^0$  has the properties

$$(\forall t \in \mathbf{R}_+) (\varphi(t) < t \wedge \limsup_{z \rightarrow t+0} \varphi(z) < t)$$

Then  $T$  has a fixed point  $\xi \in X$ .

**Proof.** Let  $x = x_0$  be an arbitrary element of  $X$ , and let  $x_n = T^n x$  ( $n \in \mathbf{N}$ ). The sequence  $\{\delta[O(x, n)]\}$  is a convergent (i.e. increasing and bounded in  $\mathbf{R}_+$ ). Let  $t = \lim \delta[O(x, n)]$ . Therefore, from (C) we have

$$t = \limsup_{n \rightarrow \infty} \delta[O(x, n)] \leq \limsup_{n \rightarrow \infty} \varphi(\delta[O(x, n)]) \leq \limsup_{z \rightarrow t+0} \varphi(z) < t,$$

Before going to the theorems, we first recollect the following definitions. A mapping  $T$  of a bounded subset  $K$  of a normed space  $X$  into itself is said to have **property  $B_k$  on  $K$**  if for every closed convex subset  $F$  of  $K$ , mapped into itself by  $T$  and containing more than one element, there exist an  $x \in F$  and a positive integer  $k$  such that  $\|x - T^k x\| < \sup \{\|y - T^k y\| : y \in F\}$ .

If  $T$  is a mapping of  $K$  into itself such that for each  $x \in K$ ,  $\lim_n \delta[O(T^n x, \infty)] < \delta[O(x, \infty)]$  when  $\delta[O(x, \infty)] > 0$ , then  $T$  is said to have **diminishing orbital diameters over  $K$**  (see [2]).

It has been shown in [14] that if  $K$  has normal structure then a mapping  $T$ , having property  $\varphi_{RBS}$ -contraction on  $K$  into itself must have property  $B_k$  on  $K$  but not conversely.

Here we obtain some fixed point theorems for mappings having property  $\varphi_{RBS}$ -contraction by using certain additional hypotheses. Then we compare the notions of diminishing orbital diameters, normal structure, and property  $B_k$  (see [14]).

We are now in a position to formulate our theorem.

**Theorem 5.** (Tasković [14]). *Let  $X$  be a normed space and let  $T$  be a mapping of  $X$  into itself having the property of  $\varphi_{RBS}$ -contraction over  $X$ . Then if  $T$  has diminishing orbital diameters over  $X$ ,  $T$  has the property  $B_k$  over  $X$ .*

**Theorem 6.** *Let  $T$  be a mapping of  $K$  into itself having property generalized  $\varphi_{RBS}$ -contraction over  $K$ . Then the following statements are equivalent:*

- (a)  $T$  has property  $B_k$  over  $K$ .
- (b) For every nonempty bounded closed convex  $T$ -invariant subset  $F$  of  $K$  which contains more than one element there exists  $x \in F$  such that

$$\sup \{\|x - T^r x\| : r \in \mathbb{N}\} < \sup \{\|z - T y\| : y, z \in F\}.$$

**Proof of Theorem 5.** we have in paper [14].

**Proof of Theorem 6.** To show that (a) implies (b) it is sufficient to see that if  $x$  be the element such that  $\|x - T^k x\| < \sup \{\|y - T^k y\| : y \in F\}$  then the element  $Tx \in F$  would satisfy the hypothesis of (b) because  $\|Tx - T^r(Tx)\| \leq \|x - T^k x\|$  by the nature of  $T$ .

We now show that (b) implies (a). If possible let (a) be not true. Then there exist a nonempty bounded closed convex subset  $F$  of  $K$  which is  $T$ -invariant and contains more than one element such that, for every  $x \in F$ ,  $\|x - T^k x\| = \sup \{\|y - T^k y\| : y \in F\} = 0$ , say. Now consider  $F' = \text{Clconv}(TF)$ . For any two elements  $z, w \in F'$ , it can be easily seen that  $\|z - Tw\| < r = \|x - T^k x\|$ . Also since  $F'$  is  $T$ -invariant and is contained in  $F$ , it follows that, for every  $z \in F'$ ,  $\sup \{\|z - T^r z\| : r \in \mathbb{N}\} = \sup \{\|z - Tw\| : z, w \in F'\}$  and this is in contradiction with (b). Hence the theorem.

Throughout this paper, unless otherwise mentioned,  $X$  is a reflexive Banach space and  $K$  a nonempty bounded closed convex subset of  $X$ . And finally, at the next step we prove a very general fixed point theorem.

**Theorem 7.** Let  $X$  be a reflexive Banach space and  $K$  be a nonempty bounded closed convex subset of  $X$ . Let  $T$  be a continuous mapping of  $K$  into  $X$  such that generalized  $\varphi_{RBS}$ -contraction of  $K$  and  $T$  maps the boundary of  $K$  into  $K$ . If  $F$  be a closed convex subset of  $K$  which contains more than one element and if  $G$  be a subset of  $F$  such that  $TG \subset F$  then there exists  $x \in G$  such that  $\|x - T^k x\| < \sup\{\|y - T^k y\| : y \in G\}$ ,  $k \in \mathbb{N}$ ; then set  $I(K, T)$  is nonempty.

A theorem similar to Theorem 7. for mapping of taipy ( $K$ ) may be seen in [6].

**Proof.** Let  $\Gamma$  be the family of all closed convex subsets  $F$  of  $K$  such that  $F \cap K \neq \emptyset$  and  $T: F \cap K \rightarrow F$  obviously  $F \in \Gamma$ . If  $\{F_\alpha\}$  be any descending chain of subsets of  $\Gamma$  then the weak compactness of each  $F_\alpha \cap K$  implies that  $F \cap K$ , where  $F = \bigcap F_\alpha$ , is nonempty. Also  $T: F \cap K \rightarrow F$  because  $T: F_\alpha \cap K \rightarrow F$  for each  $\alpha$ . Hence by Zorn's lemma there exists a minimal element  $S$  in  $\Gamma$ ,  $S$  being minimal with respect to being closed, convex and such that  $S \cap K \neq \emptyset$  and  $T: S \cap K \rightarrow S$ . We may assume  $\partial_S K \neq \emptyset$  for otherwise  $S \subset K$  and  $T: S \cap K \rightarrow S$  implies  $T: S \rightarrow S$  and if  $S$  contains only one element, the theorem is obvious. If not, by property theorem there exists  $x \in S$  such that (1):  $\|x - T^k x\| = r < \sup\{\|y - T^k y\| : y \in S\}$ . Let  $P = \{x \in S : \|x - T^k x\| \leq r\}$ . If  $x \in P$ , then since  $\|Tx - T^{k+1}x\| \leq \varphi$  (diam  $\{x, Tx, T^k x, T^{k+1}x\}$ ) we have  $\|Tx - T^{k+1}x\| \leq r$  which implies  $T(P) \subset P$ . Let  $P_1 = Cl(\text{conv}(TP))$ . If  $z \in P_1$ , then any one of the following three cases may arise: (a)  $z \in TP$  and since  $TP \subset P$ , hence  $Tz \in P_1$ . (b)  $z = \sum_1^n \alpha_i Tz_i$ ,  $\alpha_i \geq 0$ ,  $\sum_1^n \alpha_i = 1$  and  $z_i \in P$ ,

$$\begin{aligned} \|z - T^k z\| &= \|\sum_1^n \alpha_i Tz_i - T^k z\| \leq \sum_1^n \alpha_i \|Tz_i - T^k z\| \leq \\ &\leq \sum_1^n \alpha_i \varphi (\text{diam} \{z, Tz_i, T^{k-1}z, T^k z\}) \leq \sum_1^n \alpha_i r \leq r, \end{aligned}$$

which implies  $z \in P$  and hence  $Tz \in TP \subset P_1$ . (c)  $z$  is a limit point of  $P_1$ , in which case by the continuity of  $T$  it follows that  $z \in P$  and hence  $Tz \in P_1$ .

Thus  $P_1$  is a closed, convex subset of  $S$  which is invariant under  $T$  and, for every element  $z \in P_1$ ,  $\|z - T^k z\| \leq r$ , which implies by (1) that  $P_1$  is a proper subset of  $S$ . This contradicts the minimality of  $S$ . Hence  $S$  contains only one element. This element is the fixed point of  $T$ , and  $I(K, T)$  is nonempty.

One can prove in the same manner the part of this theorem concerning now  $T: \partial_S K \rightarrow K$  and  $S \cap K$  contains more than one element, we will show that we arrive at a contradiction. If  $S \cap K$  contains only one element  $\xi$ , then the nonemptiness of  $\partial_S K \subset S \cap K$  implies that  $z \in \partial_S K$  and  $T: \partial_S K \rightarrow S \cap K$  implies that  $T\xi = \xi$  which proves the theorem.

Kakutani [12] have shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some nonempty compact convex subset invariant, then the family has a common fixed point in this invariant subset. The question naturally arises as to whether this is true if one considers a commutative family of continuous not necessarily linear transformations. We shall show that it is true in a rather special, but non-trivial, case, thus giving some hope that further investigation of the general question will yield positive results. The main result of this chapter is the following.

In [15] we introduced the concept of a *diametral contraction*  $T$  of a Banach space  $X$  into itself i.e. of a mapping  $T: X \rightarrow X$  such that for every  $x, y \in X$ ,

$$\|Tx - Ty\| \leq \varphi(\sup\{\|x - y\| : y \in X\}),$$

where the existing mapping  $\varphi: \mathbf{R}_+^0 \rightarrow \mathbf{R}_+^0$  with the property  $\varphi(t) \leq t$  for  $t \in \mathbf{R}_+$ .

**Theorem 8.** Let  $B$  be a Banach space and let  $X$  be a nonempty compact convex subset of  $B$ . If  $\mathcal{F}$  is a nonempty commutative family of diametral contractive mappings of  $X$  into itself, then the family  $\mathcal{F}$  has a common fixed point  $\xi$  in  $X$ .

Proof of theorem we give in [15]. In this paper we proof one fundamental Proposition.

**Remarks.** If the norm for  $B$  is strictly convex, then the above theorem is almost trivial since in this case each contraction mapping has a fixed-point set which is nonempty, compact, and convex. In the general case, however, the fixed-point set of a diametral contraction mapping is not convex. An example illustrating this fact is constructed as follows. Let  $B$  be the space of all ordered pairs  $(a, b)$  of real numbers, where if  $x = (a, b)$ , then  $\|x\| = \max\{|a|, |b|\}$ . Define  $X = \{x: \|x\| \leq 1\}$  and  $T: X \rightarrow X$  as follows: if  $x = (a, b)$ , then  $T(x) = (|b|, b)$ . It is easily shown that  $T$  is a diametral contraction mapping and that  $x = (1, 1)$  and  $y = (1, -1)$  are fixed points for  $T$ . However,  $1/2(x + y) = (1, 0)$  is not a fixed point for  $T$ .

Now we use the following Proposition.

**Proposition 1.** (a) Let  $B$  be a Banach space and let  $M$  be a nonempty compact subset of  $B$  and let  $K$  be the closed convex hull of  $M$ . Let  $d$  be the diameter of  $M$ . If  $d > 0$ , then there exists an element  $u \in K$  such that  $\sup\{\|x - u\| : x \in M\} < d$ .

(b) Let  $X_0$  be a nonempty convex subset of a Banach space and let  $T$  be a diametral contraction mapping of  $X_0$  into itself. If there is a compact set  $M \subset X_0$  such that  $M = \{T(x) : x \in M\}$  and  $M$  has at least two points, then there exists a nonempty closed convex set  $K_1$  such that  $T(x) \in K_1 \cap X_0$  for all  $x \in K_1 \cap X_0$  and  $M \cap CK_1 \neq \emptyset$ .\*)

**Proof (a).** Since  $M$  is nonempty and compact, we may find  $x_0, x_1 \in M$  such that  $\|x_0 - x_1\| = d$ . Let  $M_0 \subset M$  be maximal so that  $M_0 \supset \{x_0, x_1\}$  and  $\|x - y\| = 0$  or  $d$  for all  $x, y \in M_0$ . Since  $M$  is compact and we are assuming  $d > 0$ ,  $M_0$  must be finite. Let us assume  $M_0 = \{x_0, x_1, \dots, x_n\}$ . Now let us define

$$u = \sum_{k=0}^n (1+n)^{-1} x_k \in K.$$

Since  $M$  is compact, we can find  $y_0 \in M$  such that  $\|y_0 - u\| = \sup\{\|x - u\| : x \in M\}$ . Now

$$\|y_0 - u\| \leq \sum_{k=0}^n (1+n)^{-1} \|y_0 - x_k\| \leq d$$

\*)  $CK_1$  is the complement of  $K_1$

because  $\|y_0 - x_k\| \leq d$  for all  $k = 0, 1, \dots, n$ . Therefore, if  $\|y_0 - u\| < d$ , then we must have  $\|y_0 - x_k\| = d > 0$  for all  $k = 0, 1, \dots, n$ , which means that  $y_0 \in M_0$  by definition of  $M_0$ . But then we must have  $y_0 = x_k$  for some  $k = 0, 1, \dots, n$ , which is a contradiction. Therefore,  $\|y_0 - u\| = d$ .

Proof (b). If we take  $K$  as the closed convex hull of  $M$ , then by (a) there exists an element  $u \in K$  such that  $d_1 = \sup \{\|x - u\| : x \in M\} < d$ , where  $d$  is the diameter of  $M$ . Since  $M$  has at least two points, we have  $d > 0$ , so that our use of (a) is valid.

For each  $x \in M$  let us define  $U(x) = \{y : \|y - x\| \leq d_1\}$ . Since  $u \in U(x)$  for each  $x \in M$ , we have  $K_1 = \bigcup_{x \in M} U(x) \neq \emptyset$ . It is clear that  $K_1$  is closed and convex. For any  $x \in K_1 \cap X_0$ , and any  $z \in M$  we have  $x \in U(z)$ , i.e.  $\|x - z\| \leq d_1$ . Since  $M = \{T(y) : y \in M\}$ , there must exist  $y \in M$  such that  $z = T(y)$ . Since  $T$  is a diametral contraction mapping, we have

$$\begin{aligned} \|T(x) - z\| &= \|T(x) - T(y)\| \leq \varphi(\sup \{\|x - y\| : y \in M\}) \\ &\leq \sup \{\|x - y\| : y \in M\} = d_1 \end{aligned}$$

i.e.  $T(x) \in U(z)$ . Since this is true for any  $z \in M$ , we have  $T(x) \in K_1 \cap X_0$ . We have shown that  $T(x) \in K_1 \cap X_0$  for all  $x \in K_1 \cap X_0$ .

Since  $M$  is compact, there exist  $x_0, x_1 \in M$  such that  $\|x_0 - x_1\| = d > d_1$ . Thus, we see that  $x_1$  does not belong to  $U(x_0) \supset K_1$ , i.e.  $x_1 \in M \cap CK_1 \neq \emptyset$ .

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