

PARTIALLY ORDERED SETS AND SOME FIXED POINT THEOREMS

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Abstract. In this paper we formulate and prove an elementary fixpoint theorem which holds in arbitrary partially ordered sets. In the following section, we give various applications (and extensions) of this results in the theory of simply ordered sets. With such an extension, a very general fixed point theorem is obtained which includes a recent result of the author, an also contains, as special cases, some results of A. Tarski, Đ. Kurepa, Abian, Metcalf and Payne, Höft, Smitson, Brown, and many others.

1. Introduction, results, and commentary

A number of papers giving fixed point theorems for decreasing, or increasing functions on partially ordered sets have appeared in the last twenty years. For example there is a Tarski's classical result for increasing functions on a lattice [9] and Davis subsequent proof of the converse [3]. Abian and Brown extended Tarski's theorem to more general partially ordered sets [2], and in [8] Smitson further, extended the result of [2] and [3]. Ward [11] used the fixed point property for increasing functions to characterize compactness of the interval topology on semi-lattices, and Tasković [10] used the fixed point property for increasing functions to characterize semi-completeness in partially ordered sets. Abian [1] gave a sufficient condition for decreasing functions on totally ordered sets to have fixed points, and then Metcalf and Payne [6] and Đ. Kurepa [5] extended Abian's result to include functions which were neither decreasing nor increasing.

In the following, (P, \leq) will denote a nonempty partially ordered set P with partial order \leq . If $f: P \rightarrow P$ is any mapping of P into P , we desing by $I(P, f)$ the set of all invariant points of P relative to f ; i. e. $I(P, f) := \{x \mid x \in P \wedge fx = x\}$. For any $f: P \rightarrow P$ it is natural to consider the following sets

$$P^f := \{x \mid x \in P \wedge x \leq fx\}, P_f := \{x \mid x \in P \wedge fx \leq x\}.$$

An ordered set (P, \leq) is said to be *right(left) conditionally complete* if every non void subset M of P which is bounded from above(below) determines its own supremum(infimum). *Conditionally complete* means to be both left and right *conditionally complete*. One proves easily that right conditional

completeness, left conditional completeness and conditional completeness are three properties which are pairwise equivalent.

Up to now, all theorems contain as an assumption the following condition: For the set $I(P, f)$ to be nonempty, it is sufficient for $f: P \rightarrow P$ to be monotone (increasing or decreasing) and P to be complete. Now we prove the theorems which does not assume these conditions. Especially, when P is totally ordered set in question, it is interesting in some way a characterisation of the sets. Namely the following assertion is valid.

Theorem 1 (Tasković [10]) *Let (P, \leq) be a totally ordered set by the order relation \leq , and $f: P \rightarrow P$ a decreasing mapping. Then the following equivalence holds:*

$$(1) \quad (\forall t \in P) \min \{t, f(t)\} \leq \xi \leq \max \{t, f(t)\}$$

$$\Leftrightarrow \xi = \min P_f \vee \xi = \max P^f.$$

From this assertion as a direct consequence follows that:

1) The number of points $\xi \in P$ with characteristic (1) can be 0, 1, or 2. Besides that:

2) Every of these cases can be realized.

3) Especially if P in the meaning of order is every where dense set of points then the number of points with characteristic (1) is 0 or 1, and

4) if set P has characteristic of density (= that for every Dedekind's cross section lower class has maximum or upper class has minimum) the number of points are 1 or 2.

5) If $\xi \in P$ is the fixed point of mapping $f: P \rightarrow P$ then ξ it is the point with characteristic (1).

Further, we prove a very general fixed point theorem which generalizes great number of known results.

Theorem 2. *Let (P, \leq) be a partially ordered set and f a mapping from P into P such that:*

(A) *The supremum of the set P^f exists. If we denote it by $s = \sup P^f$ then $f(s) \leq s$.*

(B) *$f(s)$ is majorant for the set $f(P^f)$.*

Then:

(2.1.) *The set $I(P, f)$ is nonempty,*

(2.2) *Neither of the conditions (A), (B) can be deleted if (2.1) is to be valid,*

(2.3) *Dually, if the infimum of the set P_f exists denoted by $I_m = \inf P_f$, and if $I_m \leq f(I_m)$, and if $f(I_m)$ is minorant for the set $f(P_f)$; then the set $I(P, f)$ is a nonempty*

(2.4) *There does not exist $f: P \rightarrow P$ for which condition (B) is valid as well as $x < y \Rightarrow fy < fx$ ($x, y \in P^f$).*

Theorem 3. *Let (P, \leq) be a partially ordered set and f a mapping from P into P such that (A) and:*

(C) *$f(\sup P^f) = \inf f(P^f)$,*

(D) *$x, y \in P^f \wedge x \leq y \Rightarrow fy \leq fx$.*

(G) *P^f is a totally ordered set.*

Then:

(3.1) The set $I(P, f)$ is nonempty.

(3.2) Neither of the conditions (A), (C), (D), (G) can be deleted if (3.1) is to be valid.

(3.3) Dually, if the infimum I_m of the totally ordered set P_f exists and if: $I_m \leq f(I_m)$, $f(\inf P_f) = \sup f(P_f)$, and $x \leq y \Rightarrow fy \leq fx$ ($x, y \in P_f$); then the set $I(P, f)$ is a nonempty set.

(3.4) There does not exist $f: P \rightarrow P$ for which condition $f(\sup X) = \inf f(X)$, $X \subset P_f$, is valid as well as $x < y \Rightarrow fx < fy$ ($x, y \in P_f$).

Some remarks

1) The conditions (B) and (C) are not comparable, which is illustrated by the following examples for $P = [0, 1]$ and $f: [0, 1] \rightarrow [0, 1]$, defined geometrically by:

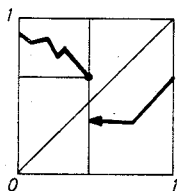


Fig. 1

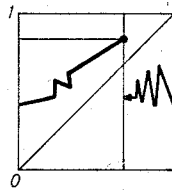


Fig. 2

In the fig. 1. the condition (C) is fulfilled and (B) not, in the fig. 2. the opposite is valid.

2) The condition (C) in theorem 3. can be weakened the relation $\inf f(P_f) \leq f(\sup P_f)$, (see proof of theorem).

3) When the conditions (B) i. e. (C) in Theorem 2. i. e. Theorem 3. are replaced by the condition: The set P_f has a maximum, then the set $I(P, f)$ is also nonempty.

4) The conditions (B) and: the existence of the supremum of set $(P_f, <)$ are not sufficient for the non-emptiness of the set $I(P, f)$, (see. Fig. 1).

5) This trivial proposition is correct:

Proposition 1 Let P be a nonempty left complete ordered set, $f: P \rightarrow P$ and $(\forall x \in P) f(x) \leq x$. Then the set $I(P, f)$ is nonempty. The theorem obtained by duality is also valid. The weakening $(\forall x \in P) fx \leq x$ of the condition $(\exists x \in P) fx \leq x$ relaxed that the set $P_f \neq \emptyset$, but this is not sufficient for the nonemptiness of the set $I(P, f)$.

In this meaning we can quote the following consequence, interesting for the set $P = \mathbf{R}_+ := [0, \infty)$, where is the relation order \leq usual number order \leq .

Proposition 2. (Tasković [10]) Let the mapping $\varphi: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ have the properties $(\forall t \in \mathbf{R}_+) \varphi(t) \leq t$ and $\limsup_{z \rightarrow t+0} \varphi(z) < t$ for $t \in \mathbf{R}_+$. If the sequence (x_n) of nonnegative real numbers satisfy the condition $x_{n+1} \leq \varphi(x_n)$, $n = 1, 2, \dots$, then the sequence (x_n) tends to zero. The velocity of this convergence is not necessarily geometrical.

6) Further let us quote one more theorem which directly follows from our theorem 2.

Proposition 3. (Corollary of Theorem 2.) Let (P, \leq) be a partially ordered set and f a mapping from P into P such that:

(E) $x, y \in P \wedge x \leq y \Rightarrow fx \leq fy$.

(F) The supremum of the set P_f exists, and if we denote it by $s = \sup P_f$ then $s \in P_f$.

Then:

(4.1) The set $I(P, f)$ is nonempty.

(4.2) Neither of the conditions (E), (F) can be deleted if (4.1) is to be valid.

(4.3) Dually, if the infimum of the set P_f denoted by $I_m = \inf P_f$ exists and $I_m \in P_f$, and if (E), then the set $I(P, f)$ is a nonempty.

Proof. Since $f: P \rightarrow P$ is an increasing mapping, the condition (B) is satisfied, and also the condition (A) as element $s = \sup P_f$ exists and also $f(s) \leq s$. This proves the corollary (proposition 4.1). Now we prove that the condition (E) and (F) cannot be removed. We show that by the following examples.

Example 1. Let $P = [0, 1)$, and define $f: P \rightarrow P$ by $f(x) = (x+1)/2$ for $x \in P$. Then condition (E) is satisfied, but condition (F) is not satisfied. Furthermore, f does not have a fixed point.

Example 2. In the fig. 2 the condition (F) is fulfilled and (E) is not. Also, f does not have a fixed point.

In an analogous way the dual results of proposition 3 can be proved.

2. Some corollaries

Now we shall apply the previous results through the following consequence. They bring into connection the results (sufficient conditions) which were obtained if the set $I(P, f)$ is nonempty.

Corollary 2.1 (Tarski [9]) Let the lattice P be completely ordered and $f: P \rightarrow P$ an increasing mapping. Then the set $I(P, f)$ is nonempty.

Corollary 2.2 (Kurepa [4]) Let P be the ordered set and $f: P \rightarrow P$ increasing mapping if P is left complete and if the set P_f is nonempty then the set $I(P, f)$ is nonempty ordered and left complete.

Corollary 2.3 (Kurepa [5]) Let P be a nonempty right conditionally complete ordered set and f a decreasing selfmapping of P such that for at least one member $x \in P$ we have

$$x \leq fx \vee fx \leq x, \text{ i. e. } \neg (\forall x \in P, x \parallel fx).$$

Let us assume that

1) $f(\sup) = \inf f$,

2) Each point P_f is comparable with each point of P_f ,

3) If $s := \sup P_f \in P$ exists then $f(s) \leq s$.

Then the set $I(P, f)$ is nonempty and $f(s) = s = \inf P_f$.

In the following (P, \leq) will denote a nonempty partially ordered set P with partial order \leq . A subset A of P is a *toset* just in case A is totally ordered. For $x \in P$ and $A \subset P$, define $L(x) = \{y: y \leq x\}$, $M(x) = \{y: x \leq y\}$ and $M(A) = \bigcup \{M(x): x \in A\}$.

A partially ordered set (P, \leq) is a *mod* if and only if the following hold:

- 1) For all $x, y \in P$ $\sup \{x, y\}$ exist,
- 2) For all $x \in P$, $L(x)$ is a toset,
- 3) Each nonempty subset of P which is bounded above (below) has a supremum (infimum) in P ,
- 4) If $x < y$, then there is a $z \in P$ such that $x < z < y$.

A function $f: P \rightarrow P$ is *nonoscillatory from above* if and only if for each nonmaximal x and maximal toset $A \subset M(x) \setminus \{x\}$, $\bigcap \{f([x, u]): u \in A\} = \{f(x)\}$. The function f is *nonoscillatory from below* if and only if for each nonminimal x , $\bigcap \{f([u, x]): u < x\} = \{f(x)\}$.

Corollary 2.4. (Metcalf and Payne [6]) *Let P be a totally ordered mod. Suppose that $f: P \rightarrow P$ is a function satisfying:*

- (1) *If $x \leq y$ and $f(y) \leq f(x)$, then $[f(y), f(x)] \subset f([x, y])$,*
- (2) *The function f is either nonoscillatory from above or from below.*
- (3) *There exist $a, b \in P$ such that $a \leq b$, $a \leq f(a)$, and $f(b) \leq b$.*

Then f has a fixed point.

We next demonstrate that the following condition introduced by Abian [1] is a form continuity.

Abian's condition. Let $f: P \rightarrow P$ where P is a mod. If $A \subset P$ is a toset, then $f(\inf A) = \sup f(A)$ and $f(\sup A) = \inf f(A)$ whenever both sides of the, equalities exist.

Corollary 2.5. (Abian [1]) *Let $f: P \rightarrow P$ where P is a totally ordered mod. If f is decreasing and satisfies Abian's condition, then f has a fixed point.*

Corollary 2.6. (H. and M. Höft [7]) *Suppose the partially ordered set P satisfies:*

- (a) *For every order-preserving map $f: P \rightarrow P$ there exists $x \in P$ such that x and $f(x)$ are comparable, i. e. either $x \leq f(x)$ or $f(x) \leq x$.*
- (b) *Every non-empty chain of P has a supremum and an infimum.*

Then P has the fixed point property, i. e. if every order-preserving map $f: P \rightarrow P$ has a fixed point.

Remark. It is interesting that our theorem can be applied in the case of finite partially ordered sets. Namely, using the theorem and Höft's [7] theorem on lexicographical representation of partially ordered sets, it can be shown that every increasing function partially ordering in Fig. 3 has a fix point property.

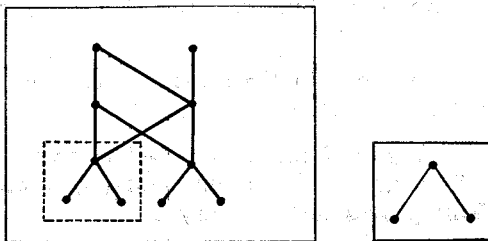


Fig. 3

3. Proofs of Theorems and corollaries

Proof of Theorem 1.2.1. The set P^f being, by assumption, nonempty the point $s = \sup P^f$ exists. We have the following chain of relations:

$$s = \sup P^f \leq \sup f(P^f) \leq f(\sup P^f) \leq \sup P^f = s,$$

and thus $s \in I(P, f)$, i. e. set $I(P, f)$ is a nonempty.

2.2. Now prove that the conditions (A) and (B) cannot be removed. We show that by the following examples.

Example 3. Let P be the set $[0, 2]$ and define $f: [0, 2] \rightarrow [0, 2]$ by $f(x) = 1$ for $x \in [0, 1]$ and $f(x) = 0$ for $x \in [1, 2]$, where P is a totally ordered set by ordinary ordering \leq . The condition (A) is satisfied, but condition (B) is not satisfied. Furthermore, f does not have a fixed point.

Example 4. Let $P = [0, 2]$, and define $f: P \rightarrow P$ by $f(x) = 2$ for $x \in [0, 1]$ and $f(x) = 1$ for $x \in (1, 2]$. Then condition (B) is satisfied, but condition (A) is not satisfied. Furthermore, f does not have a fixed point.

2.3. By dual considerations one proves the part of Theorem which concerns the point $s = \sup P^f$. It suffices to make the following substitutions:

$$\sup \rightarrow \inf, P^f \rightarrow P_f, \leq \rightarrow \geq.$$

2.4. Let $f: P \rightarrow P$ exist and satisfies (B) and assume that condition $x < y \Rightarrow fy < fx$ ($x, y \in P^f$) be satisfied. When f is an increasing mapping, the condition $\sup f(X) \leq f(\sup X)$ is satisfied, what is a contradiction.

Proof of Theorem 3. 3.1. We have the following chain of relations:

$$s = \sup P^f \leq \inf f(P^f) = f(\sup P^f) \leq \sup P^f = s,$$

and other $s \in I(P, f)$.

Example 5. Let $P = [0, 1]$, and define $f: P \rightarrow P$ by $f(x) = (x+1)/2$ for $x \in [0, 1]$ and $f(1) = 1/2$. Then condition (C) is satisfied ($f(\sup P^f) = f(1) = 1/2 = \inf f(P^f)$), condition (G) is satisfied, and condition (A) is satisfied ($f(\sup P^f) = f(1) = 1/2 < 1 = \sup P^f$), but condition (D) is not satisfied. Furthermore, f does not have a fixed point.

Example 6. Let the set $P = \{a, b, c, d, e, g, g_n (n \in \mathbb{N})\}$ be ordered by the relation order \leq so that $a \leq c, a \leq d, b \leq e, b \leq c, g \leq e, g \leq d, g \leq c, g_1 \leq g, g_{n+1} \leq g_n (n \in \mathbb{N})$, when the elements a, g, b are incomparable and also the elements c, d, e are incomparable; and define $f: P \rightarrow P$ by: $f(a) = d, f(b) = e, f(d) = f(e) = f(c) = g, f(g) = g_1$ and $f(g_n) = g_{n+1} (n \in \mathbb{N})$.

The condition (A) is satisfied ($P^f = \{a, b\}, f(\sup P^f) = f(c) = g \leq s = \sup P^f = \sup \{b, a\} = c$), condition (C) is satisfied ($f(\sup P^f) = f(c) = g = \inf f(P^f) = \inf \{d, e\} = g$) and condition (D) is satisfied, but condition (G) is not satisfied ($a, b \in P^f$ are incomparable); and $f: P \rightarrow P$ has not fixed point.

In an analogous we prove that the condition (A) and (C) cannot be removed.

Remark. One can prove in the same manner the part of this theorem concerning for 3.2., 3.3. and 3.4.

Proof of Corollary 2.1. Since $f: P \rightarrow P$ is an increasing mapping, the condition (B) is satisfied, and also the condition (A) as element $x = \max P \in P^f$ and also $f(x) \leq x$. This proves the corollary 1.

In an analogous way the results of KUREPA [4], [5], ABIAN [1], METCALF and PAYNE [6] and HÖFT H., HÖFT M. [7], follow as corollaries of our theorems 2 and 3. The proofs are more than obvious. We shall prove only the corollary 6.

Proof of Corollary 2.6. Since f is an increasing mapping, the condition (B) of Theorem 2. is satisfied. We shall prove the condition (A) of Theorem 2. The system of chains L for which $t \in L$ implies $ft \in L$ and $t \leq ft$ contains the nonempty chain $T = \{f^k(t) | k = 0, 1, 2, \dots\}$, and therefore contains a maximal chain M (Zorn's Lemma). By assumption, $x = \sup M \in P$ exists. Since M satisfies condition $t \in L$ implies $f(t) \in L$ and $t \leq f(t)$ we have $t \leq f(t) \leq f(x)$ for all $t \in M$, so that $x \leq f(x)$; and x is not in M , then the chain $M \cup T$ property contains M , and satisfies $t \in L$ implies $ft \in L$ and $t \leq f(t)$, in contradiction to the maximality of M . Therefore, $x \in M$ and also $fx \in M$, hence $f(x) \leq x$. This proves the corollary.

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