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OSCILLATION OF SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS

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Abstract. An oscillation criterion is given for the second order nonlinear equation $\Delta(r_{n-1}\Delta u_{n-1})+a_n f(u_n)=0$, which generalizes McCarthy's result.

In this note we are concerned with the solutions of the second order nonlinear difference equation

$$\Delta(r_{n-1}\Delta u_{n-1})+a_n f(u_n)=0 \quad n=1, 2, \dots, \tag{1}$$

where Δ is the forward difference operator i.e. $\Delta u_n=u_{n+1}-u_n$, $\{r_n\}_0^\infty$, $\{a_n\}_0^\infty$ are the real sequence, $f: R \to R$.

By a solution of (1) we mean a real sequence $\{u_n\}_0^\infty$ $(u_n \not\equiv 0)$, satisfying equation (1) for $n=1, 2, \dots$.

A solution $\{u_n\}_0^\infty$ is called nonoscillatory if there exists $n_0 \geqslant 0$, such that for every $n \geqslant n_0$ either $u_n>0$ or $u<0$; otherwise it is said to be oscillatorv.

In the paper [3] P. J. McCarthy considered the second order linear difference equation

$$\Delta(r_{n-1}\Delta u_{n-1})+a_n u_n=0, \quad n=1, 2, \dots \tag{2}$$

and obtained the following resut:

Theorem. *If* $r_n>0$, $(n \geqslant 0)$, $\{r_n\}_0^\infty$ *is a bounded sequence and* $\sum^\infty a_n=\infty$, *then all non-trivial solutions of* (2) *are oscillatory.*

The purpose of this note is to prove a similar result for (1) which when applied to (2) generalizes McCarthy's result, namely we replace the requirement that $\{r_n\}_0^\infty$ is bounded by the weaker conditon $\sum^\infty r_n^{-1}=\infty$. Our Theorem below is the discrete analogue of the Bhatia theorem [1] for nonlinear differential equations of second order and in particular Leighton's theorem [2].

Other oscillation criteria for nonlinear difference equations of the form (1), where the coefficient $\{a_n\}_0^\infty$ is assumed to be eventually nonnegative, are contained in [4].

Theorem. *Let the following conditions are valid:*

1° *$f: R \to R$ is nondecreasing, $sf(s) > 0$ for $s \neq 0$,*

2° $r_n > 0$, $(n \geq 0)$, $\sum^{\infty} r_n^{-1} = \infty$,

3° $\sum^{\infty} a_n = \infty$,

then every solution of (1) *is oscillatory.*

Proof. Suppose there exists a nonoscillatory solution $\{u_n\}$ of (1) and let $u_n > 0$ for $n \geq n_1 - 1 \geq 0$. For $n \geq n_1$ we set

$$q_n = \frac{r_{n-1} \Delta u_{n-1}}{f(u_{n-1})}.$$

Then from equation (1) we obtain

$$\Delta q_n = -a_n - \frac{r_{n-1} \Delta u_{n-1} \Delta f(u_{n-1})}{f(u_{n-1}) f(u_n)}.$$

We note by 1° that the second term on the right is nonnegative for $n \geq n_1$, hence

$$\Delta q_n \leq -a_n, \quad n \geq n_1. \tag{3}$$

Summing up both sides of (3) from n_1 to n we get

$$q_{n+1} \leq q_{n_1} - \sum_{i=n_1}^{n} a_i \to -\infty \text{ as } n \to \infty.$$

Thus $q_n < 0$ for $n \geq n_2 \geq n_1$ which implies that Δu_n is negative for large n.

From 3° it follows there exists $n_3 \geq n_2$, such that

$$\sum_{i=n_3}^{n} a_i \geq 0 \text{ for } n \geq n_3. \tag{4}$$

Summing up both sides of (1) from n_3 to n we have

$$\sum_{i=n_3}^{n} \Delta(r_{i-1} \Delta u_{i-1}) = -\sum_{i=n_3}^{n} a_i f(u_i),$$

and according to summation by parts formula we may write

$$r_n \Delta u_n - r_{n_3-1} \Delta u_{n_3-1} = -f(u_n) \sum_{i=n_3} a_i + \sum_{i=n_3}^{n-1} \Delta f(u_i) \sum_{k=n_3}^{i} a_k.$$

Since $u_{n+1}<u_n$ for $n\geqslant n_2-1$, then by 1° and (4) it follows that

$$r_n \Delta u_n \leqslant r_{n_3-1} \Delta u_{n_3-1} < 0,$$

which implies

$$\Delta u_n \leqslant r_{n_3-1} \Delta u_{n_3-1} r_n^{-1} \quad \text{for } n\geqslant n_3, \tag{5}$$

From (5), using 2°, we conclude that $u_n \to -\infty$ as $n\to\infty$, but this contradicts the fact that $\{u_n\}$ is eventually positive. A similar argument can be used in the case of an eventually negative solution. Thus the proof is complete.

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