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GRAPHS WHICH ARE SWITCHING EQUIVALENT TO THEIR COMPLEMENTARY LINE GRAPHS I

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In this paper we will find all connected graphs which are switching equivalent to their complementary line graphs. The notion of switching equivalency is taken here in Seidel's sense, see [1], while for some facilities, we also introduce some conventions from [2].

A partition of the vertex set of a graph G into two (disjoint) subsets (one of which may be empty) will be represented as a colouring c of vertices by two colours (say black and white) such that the vertices from the same subset are coloured by the same colour. The graph G together with its colouring c will be denoted by G_c. Switching a graph G with respect to a colouring c (or partition c) means deleting all edges between black and white vertices in G_c and introducing a new edge between a black and a white vertex whenever they were nonadjacent in G_c. The graph obtained after switching will be denoted by $\mathcal{S}(G_c)$. Graphs G and H are switching equivalent if $H=\mathcal{S}(G_c)$ for some colouring c. Switching relation $\sim$ is an equivalence relation in the set of graphs and this enables us to speak about switching classes of graphs.

Following [2] we can also say that we are here, in fact, solving the "generalized" graph equation $G\sim\overline{L(G)}$ or, what is the same, $L(G)\sim\overline{G}$. The "ordinary" graph equations were considered in many other papers from the literature and with them the isomorphism relation $\cong$ or rather $=$ was taken as equality, while the solutions were graphs (determined up to isomorphism). Since all graph operations considered in this paper are defined in the set of graphs but not in the set of switching classes, we must consider the unknow G in our equation rather as a graph than as a switching class, which explains the generalized concept.

In [3] the ordinary graph equation $L(G)=\overline{G}$ was solved. It has only two solutions, namely C_5 and $C_3\circ K_1$*. These two solutions are also solutions of our generalized equation and they can be regarded as ordinary ones. All other solutions are exceptional, and with them $\mathcal{S}(G_c)=\overline{L(G)}$ for some colouring c

* o denotes the corona of graphs.

while $L(G) \neq \overline{G}$. Hence, when we find a graph G which is a solution we shall prefer to indicate a colouring c (which need not be unique) so that ${}_c\mathcal{S}(G_c) = \overline{L(G)}$ holds.

Throughout this paper we shall follow the terminology from [4]. Some unusual notations will be given here.

Suppose G is a graph while V is its vertex set. If $U \subseteq V$, then $G(U)$ denotes an induced subgraph of G generated by U. Since we are dealing also with graphs whose vertices are coloured (black and white) we shall denote by $V_B(V_W)$ the subsets of all black (white) coloured vertices of a graph whose vertex set is V. The fact that H is just an induced subgraph of G will be written $H \subseteq G$, while otherwise we put $H \leqslant G$. If c is some colouring of H then it is proper if any two adjacent vertices are coloured by different colours.

Assume now that $v_1, v_2, \ldots, v_p$ are vertices of some cycle C_p such that v_i adj v_{i+1} $(i = 1, 2, \ldots, p)$ and summation is taken mod p. We define a graph $C(a_1, a_2, \ldots, a_p)$ as follows: we take a cycle C_p as above and to each v_i we add a_i pendant edges.

In solving our equation we shall use the so called growing method. We observe one detail of a graph being searched (possiblly its induced subgraph) and step by step we add to it new vertices (together with edges) in order to construct it. This process of growing is, of course, under some control. If starting from one step some vertex does not have any more neighbours we shall call it saturated.

The basic tool for our further purposes should be the following lemma which is a straightfoward consequence of Beineke's theorem for line graphs.

L e m m a 1*. If ${}_c\mathcal{S}(G_c)$ is a complement of a line graph, then G_c does not contain as "colour" induced subgraph any of the following graphs of Fig 1.

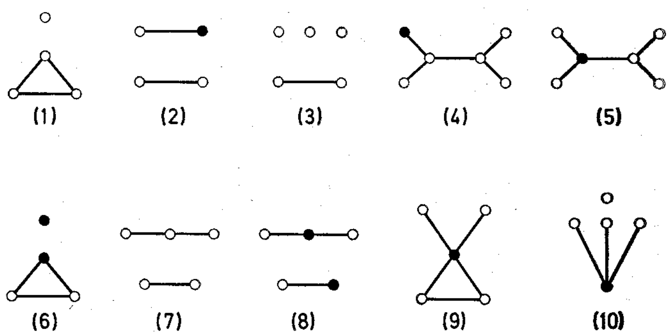

Fig. 1-

The following lemmas are also easy for proving and we only quote them.

* Of course, the colours in the above giaphs may be mutually interchanged.

Lemma 2. If $\mathscr{S}(G)=\overline{L(G)}$, then any of the graphs of Fig. 2 when appearing as colour induced subgraph in G_c imply $C_4 \leqslant G$.

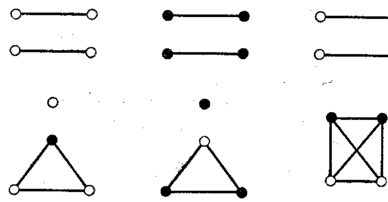

Fig. 2

Lemma 3. If G is connected and $G \sim \overline{L(G)}$, then G is unicyclic.

Lemma 4. If $K_3 \nsubseteq G$ and $\mathscr{S}(G_c)=\overline{L(G)}$, then C_4 appears in G as an induced subgraph equally as $2K_2$ in G_2 provided that both copies of K_2 are coloured in the same way.

Lemma 5. If $K_3 \nsubseteq G$ and $\mathscr{S}(G_c)=\overline{L(G)}$, then G_c does not contain as colour induced subgraph any of the following graphs of Fig. 3.

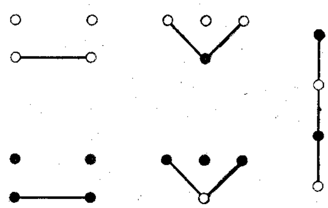

Fig. 3

Lemma 6. If $\mathscr{S}(G_c)=\overline{L(G)}$, then $\Delta(G)=\max\limits_{H_c}\{p(H_c)\}$ where H_c is an induced subgraph of G_c which is coloured properly and isomorphic to either nK_1 or $K_{m,n}$.

Note when G is connected then H can be only mK_1, $K_{1,n}$ and C_4.

Now, on the basis of Lemma 2 we can begin our growing process. Namely, we can start even from a cycle and grow it toward solution. The great facility in connected case is the fact that each added vertex can be adjacent only to one vertex already existing in the previous step.

Theorem 1. *If G is connected and $g(G)=3$, then $G \sim \overline{L(G)}$ implies G is equal to $C(m, n, 1)$ $(m, n \geqslant 0)$.*

Proof. The vertices of the unique triangle $T(=uvw)$ may be regarded in G_c as follows:

(i) all three are white;

(ii) two are white while one is black.

Case (i): Concerning only white vertices, (1)* directly implies $G(V_W)=$ $=C(a, b, c)$. From (3) we get $a+b+c \leqslant 3$ while due to (2) all white vertices out of T are saturated.

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* In further text we shall refer to Lemma 1 only by forbidden coloured graphs of Fig. 1.

Now, suppose there exists a black vertex at distance $k>1$ from the nearest vertex of T (see Fig. 4). Clearly, $v_1, v_2, \ldots, v_k$ are all black, while due to (2) $k\leqslant 3$.

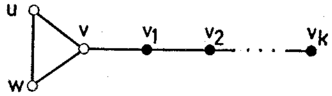


Fig. 4

Assume first $k=3$. If all black vertices (except v_2) are adjacent to v_2, then v_2 would be isolated in $\overline{\mathcal{S}(G_c)}$. Hence there must exist in G_c a black vertex, say x, nonadjacent to v_2 but adjacent to some vertex from a graph of Fig. 4. Due to (2) x could be adjacent only to v_1 and by Lemma 2 it is saturated. Further, v_2 is also saturated due to (4). Even v_1, due to Lemma 2 becomes saturated if just one vertex analogue to x is added. Since $K_5\subseteq\mathcal{S}(G_c)$ (observe T and an edge v_2v_3) we have $5K_2\leqslant G$, thus implying $a=b=c=1$. Therefore only two graphs appear as possible solutions and we eliminate them by direct checking.

Now let $k=2$. In this case since $\overline{\mathcal{S}(G_c)}$ has to be connected there must exist in G_c a black vertex, say again x, nonadjacent to v_1 but adjacent to some vertex from a graph of Fig. 4. Due to (2) x could be adjacent only to v and by Lemma 2 it is saturated. If y is black and adjacent to v, then due to (3) and (5) there are no more black vertices. The resulting few graphs cannot be solutions due to Lemma 6, for example. So except x (and v_1) all other black vertices are neighbours of v_1. If v_1 has $s\geqslant 3$ black neighbours, there are no solutions because of $\Delta(L(G))\geqslant s+3>s+2\geqslant\Delta(\mathcal{S}(G_c))$; otherwise when $s<2$ the same follows by direct checking.

Hence let $G=C(a, b, c)$. If we delete vertices u, v, w from $\mathcal{S}(G_c)$, then we get a graph $K_{s,t}$ $(s, t\geqslant 0)$. After complementation the latter graph becomes $K_s\cup K_t$. This means that in G there exist 3 mutually nonadjacent edges which can be removed so that the line graph of the last graph is equal to $K_s\cup K_t$. This is possible only if $a=b=c=1$. i.e. $G=C(1, 1, 1)$. Of course, the last graph is a solution (see Fig. 6 a) but here it can be viewed only as ordinary solution.

Case (ii): By using (6) and Lemma 2 it follows directly $G=C(a, b, c)$. Now, if we delete the vertices u, v, w from $\mathcal{S}(G_c)$ then we again get a graph $K_{s,t}$ $(s, t\geqslant 0)$ while its complement is $K_s\cup K_t$. This means that in G there exists a subgraph (not necessarily induced) isomorphic to P_3 whose lines can be deleted from G so that the line graph of resulting graph is equal to $K_s\cup K_t$. The latter is possible only if at least one integer out of a, b, c takes the value 0 or 1.

Assume first, say c, is equal to 0. Then there exists in $L(G)$ a vertex adjacent to all others. Hence there must exist in G_c a vertex, say x, adjacent to all vertices of opposite colour but nonadjacent to any vertex of its colour. If x is on T it must coincide with a black vertex of T, say u. But then all white vetices of G_c are adjacent to u while due to (9) there is at most one white vertex outside T. If just one white vertex outside T exists then we immediately get a solution indicated in Fig. 6 a (note $n=0$). Hence all vertices outside T are black and of course adjacent to either v or u. If a or b is equal to zero, then $\delta(L(G))=2$ while $\delta(\overline{{}_c\mathcal{S}(G_c)})=1$, implying $ab\neq 0$. From (10) it follows $a=b=1$ and this is a solution already registered but coloured differently (see Fig. 6. a) If x is not on T it can be adjacent only to u while all vertices out of T including x must be white. Due to (9) we now easily get a solution indiceted in Fig 6. a (note $m=0$).

Hence we can now take $c=1$. In this case without any analysis concerning colouring we have a solution indicated in Fig. 6. a.

This proves the theorem.

Theorem 2. *If G is connected and $g(G)=4$, then $G\sim\overline{L(G)}$ impies G is equal to $C(m, 1, n, 1)$ $(m, n\geqslant 0)$.*

Proof. The vertices of the unique quadrangle Q ($=$tuvw) may be regarded in G_c as follows:

(i) all four are white;

(ii) three are white and one is black;

(iii) two are white and two are black.

Case (i): Concerning (again) first only white vertices (2) and (7) directly imply $G(V)=C(a, b, c, d)$. From Lemma 5 it follows, say $a=c=0$ and $0\leqslant b$, $d\leqslant 1$ while due to (2) all white vertices outside Q are saturated.

Now, suppose there exists a black vertex at distance $k>1$ from the nearest vertex of Q (see Fig. 5). Clearly, $v_1, v_2, \dots v_k$ are all black, while due to (2) $k\leqslant 3$.

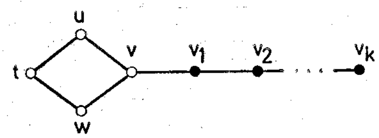


Fig. 5

If $k=3$ (see the corresponding part of Theorem 1), there must exist in G_c a black vertex x adjacent just to v_1. But then, due to Lemma 5, v_1 is saturated and also v_2, as follows from (4). From (7) it follows that we can add only one black vertex more but only to x. Now we have only a few graphs as possible solutions and they are eliminated by direct checking.

If $k=2$ (see the corresponding part of Theorem 1), there must exist in G_c a black vertex x adjacent just to v. Due to Lemma 5, v has not any more black neighbours. If v_1 besides v_2 has more black neighbours, then due to (7), x is saturated. But now due to Lemma 4, u and w each must have just one white neighbour outside Q. The resulting graph cannot be a solution since, for example, $L(G)$ has just one cut-point (observe an edge vv_1) while v and x are cut-points for $\overline{{}_c\mathcal{S}(G_c)}$. If v_2 is the only black neighour of v_1 then, due to Lemma 4 we can add only one black vertex more solely to x. Now, we have only a few graphs as possible solutions and they are eliminated by direct checking.

Hence let $G=C(a, b, c, d)$. Due to (2) and Lemma 4, say $b=d=1$. The resulting graph is a solution (see Fig 6. b) but, as can be shown. only for some other colourings.

Case (ii): Let, say u, be black. Due to (2), (7) and (8) if some edge xy is disjunct from Q then both x and y must be black and, say x is adjacent to u. Note first, that due to (2), x does not have white neighbours. Hence if all black vertices of G_c are adjacent to x, then it will be isolated in $\overline{\mathcal{S}(G_c)}$. Thus due to (2) there must exist some black vertex z which is adjacent to u. Now, having in view Lemma 5 there could exist only one black vertex more but adjacent just to z. By the same lemma, w (opposite of u) does not have white neighbours out of Q while v or t each could have at most one such neighbour. The remaining graphs are not solutions due to Lemma 6, for example.

Hence let $G=C(a, b, c, d)$. By applying Lemma 4 it immediately follows, say $b=d=1$. The resulting graph is a solution (see Fig. 6. b) but, as can be shown, only for some other colourings.

Case (iii): Now by (2) and (8) we immediately get $G=C(a, b, c, d)$ while from Lemma 4, say $b=d=1$. Now, not dealing with colourings we have a solution from Fig. 6.b.

This proves the theorem.

Theorem 3. *If G is connected and $g(G)\geqslant 5$, then $G\sim\overline{L(G)}$, implies $G=C(0, 0, 0, 0, 0)$.*

Proof. If $g(G)\geqslant 6$, then $2K_2$ is an induced subgraph of the unique cycle of G which after colouring contradicts either (2) or Lemma 3. Thus $g(G)=5$, i.e. pentagon P appears in G as an induced subgraph. If $G\neq C_5$, then $2K_2\subseteq G$ and due to (2) and Lemma 2 one copy of K_2 is white while the other is black also providing $4K_2\leqslant G$ since $K_4\subseteq\mathcal{S}(G_c)$, i.e. $4K_1\subseteq\overline{L(G)}$. If there exists in G a vertex, say x, at distance 1 from the nearest verteh of P having degree greater then one, then x and as well its neighbours must be, say black, while the remaining four vertices of P are white and also saturated. Since x could not be isolated in $\overline{\mathcal{S}(G_c)}$ there must exist in G_c a black vertex nonadjacent to x. Due to (2) it may be adjacent only to the unique black vertex of P. If deg $x>2$ we get a contradition due to (4) or if deg $x=2$ all vertices of the corresponding graph are saturated and it is not a solution as can be seen by direct checking.

Thus let $G=C(a, b, c, d, e)$. Now if $4K_2\leqslant G$ it follows that, say $a, b, c\neq 0$ regardless to d and e. But now we easily get a contradiction due to (2) or Lemma 4.

This proves the theorem.

Now we can state our main result.

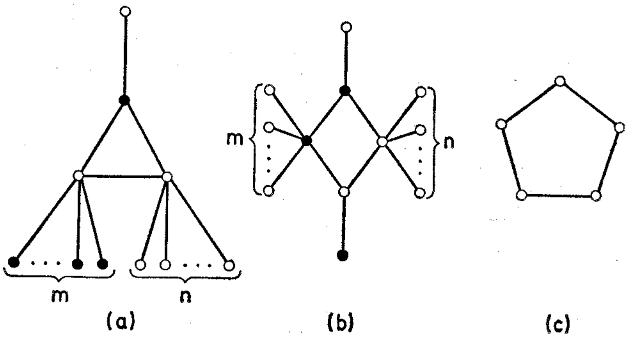
Theorem 4. *If G is connected and $G\sim\overline{L(G)}$, then G is one of the following graphs:*

(a) $C(m, n, 1)\ (m, n\geqslant 0)$;

(b) $C(m, 1, n, 1)\ (m, n\geqslant 0)$;

(c) $C(0, 0, 0, 0, 0)$.

(see also Fig. 6).



Fig, 6

The disconnected case will be treated in our future paper.

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