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ON FREE COMMUTATIVE GROUPOIDS

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1. It is well known the following equivalence lemma [5]:

$$(1) \qquad AB=CD \Leftrightarrow (A=C \wedge B=D) \vee (\exists X)(A=CX \wedge D=XB) \vee (\exists X)(C=AX \wedge B=XD)$$

where A, B, C, D, X are words in a given alphabet.

The proof can be splitted into three cases:

(i) $d(A)=d(C)$, (ii) $d(A)>d(C)$, (iii) $d(A)<d(C)$

where $d(X)$ denotes length of the word X. We recall that the equivalence (1) has many applications. For example the proof [2] that each term can be expressed by its subterms in the unique way is based on the equivalence (1). So are different asertions about equations on the word semigroups [3], [4] and the proof that the operation F defined by

$$F(X_1, X_2, \ldots, X_n)=fX_1 X_2 \ldots X_n \qquad (f \text{ is a constant symbol})$$

does not satisfy any algebraic law what implies that any Ω-algebra can be embedded into a word algebra [1].

As any free semigroup is isomorphic to the corresponding word semigroup, it is easy to conclude that in any free semigroup $(S, *)$ the wollowing equivalence

$$(2) \qquad A*B=C*D \Leftrightarrow A=C \wedge B=D \vee (\exists X)(A=C*X \wedge D=X*B) \vee (\exists X)(C=A*X \wedge B=X*D)$$

is satisfied, where A, B, C, D, X belong to S.

The equivalence (2) can also be proved directly using some well known consequences of the associative law.

Note that the operation $*$ satisfying the equivalence (2) must be associative what we are going to prove in the following lemma.

Lemma 1. *If the operation $*$ satisfies the equivalence* (2), *then it is an associative operation.*

Proof. Suppose $*$ satisfies the equivalence (2) and consider the associative law

$$(A*B)*C = A*(B*C)$$

where A, B, C are terms in $*$. We deduce the following equivalence chain:

$$\begin{aligned}
&(A*B)*C = A*(B*C)\\
&\quad\Leftrightarrow A*B = A \wedge C = B*C\\
&\qquad \vee (\exists X)\,(A*B = A*X \wedge B*C = X*C)\\
&\qquad \vee (\exists X)\,(A = (A*B)*X \wedge C = X*(B*C))\\
&\quad\Leftrightarrow A*B = A \wedge C = B*C\\
&\qquad \vee \top \qquad \text{(For } X \text{ we can choose } B)^{1)}\\
&\qquad \vee (\exists X)\,(A = (A*B)*X \wedge C = X*(B*C))\\
&\quad\Leftrightarrow \top \qquad \text{(By the tautology } (p \vee \top) \Leftrightarrow \top).
\end{aligned}$$

Thus from the assumption (2) it follows that the associative law for $*$ is equivalent to truth what yields that this law is satisfied. We mention that the groupoid $(S, *)$ satisfying (2) need not to be a free semigroup what shows the following example. Let $(S, *)$ be the free semigroup generated by $\{a, b\}$ and let a new constant c be defined by:

$$\text{Def}(c) \qquad c = a*b$$

The simigroup obtained in such a way satisfies (2) because so does each free semigroup, and it is not free because it satisfies the equality Def(c).

Our main task is to prove for the commutative operation the equivalence similar to (2). In the second part of the paper we give several applications to the word problem, solwing the equations, finding the so called lawless terms etc.

2. Let $*$ be a binary operation symbol and K she commutativ law for $*$, i.e.

$$(K) \qquad x*y = y*x$$

Further, let X, Y be terms built up from $*$, some variables and some constant symbol. First of all we give the following definitions.

(D 1) $X \equiv Y$ means that X, Y are equal as words

(D 2) $X \to_K Y$ means that Y is obtained from X by one use of the commutative law[2)]

(D 3) $X =_K Y$ means that Y is obtaing from X by the finite uses of K. In other words:

$$\begin{aligned}
X =_K Y \Leftrightarrow\ & X \equiv Y \vee X \to_K Y\\
& \vee (\exists n)\,(\exists U_1, \ldots, U_n)\,(X \to_K U_1 \wedge U_1 \to_K U_2 \wedge \cdots \wedge U_n \to_K Y)
\end{aligned}$$

where $U_1, \ldots, U_n$ are terms in $*$.

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1) The symbols $\top$, $\bot$ denote the words *true*, *false* respectevily.

2) That is one subterm of X of the form T_1*T_2 is replaced with T_2*T_1.

The definition (D 3) may be replaced with

(D 3') $X=_K Y$ iff $K\vdash_E X=Y$

where $\vdash_E$ is the sign of logical deduction in so called equational logic, that is in the logic having as axioms all formulae of the form

$$(R)\quad x=x$$

and as rules

$$(S)\ \frac{x=y}{y=x};\qquad (T)\ \frac{x=y,\ y=z}{x=z};\qquad (S)\ \frac{x_1=y_1,\ x_2=y_2}{x_1*x_2=y_1*y_2}$$

Using (D 3) or (D 3') it can easily be proved that $=_K$ is an equivalence relation compatible with $*$, i.e. that it is a congruence for $*$. We note that K in the definitions (D 1), (D 2), (D 3), (D 3') may be replaced by any set Z of algebraic laws in the given operation language O. The relation $=_Z$ is also an equivalence relation compatible with each operation in O, i.e. it is a congruence for O. Throughout the paper we shall mostly deal with the commutative law and therefore the subscripts K in relations $\rightarrow_K$, $=_K$ will often be omitted except in the case where ambiguites may arise.

In the next theorem we prove the equivalence which parallels to (2).

Theorem 1. *Let $(S, *)$ be a free commutative groupoid. Then it satisfies the following equivalence*

$$(3)\qquad A*B=C*D \Leftrightarrow (A=C\wedge B=D)\vee(A=D\wedge B=C)$$

where A, B, C, D are any terms in $*$.

Proof. As $(S, *)$ is a free commutative groupoid, it suffices to prove:

$$(4)\qquad A*B=_K C*D \Leftrightarrow (A=_K C\wedge B=_K D)\vee(A=_K D\wedge B=_K C)$$

The proof of $\Leftarrow$ *–part* follows immediately.

$\Rightarrow$ *–part*: Suppose now

$$(5)\qquad A*B=_K C*D$$

By definition (D 3) this means that $C*D$ is obtaind from $A*B$ by finite uses of the commutative law. If the number of uses is zero then

$$(A=_K C\wedge B=_K D)\vee(A=_K D\wedge B=_K C)$$

follows immediately.

Consider now the case

$$A*B\rightarrow_K C*D$$

By definition of the relation $\rightarrow_K$ the following cases[1] are possible:

(i) K is applied to A

(ii) K is applied to B

(iii) K is applied to the whole term $A*B$

[1] Generally, four cases are possible. Namely, when we apply some algebraic law Z to the whole term $A*B$, then we can use Z from right to left and from left to right, what coincides in the case o commutative law.

Using definitions D 1, D 2 the first and second cases can be writen in the form

$$A \to_K C \wedge B \equiv D, \quad A \equiv C \wedge B \to_K D$$

In the third case applying the law K to the whole term $A*B$ we obtain the term $C*D$. Therefore the following equalities

$$C \equiv B, \quad D \equiv A$$

hold. All three casses writen together yield the disjunction

$$(A \to_K C \wedge B \equiv D) \vee (A \equiv C \wedge B \to_K D) \vee (C \equiv B \wedge D \equiv A)$$

Thus we have just proved the equivalence:

$$(6) \qquad (A*B \to_K C*D) \Leftrightarrow (A \to_K C \wedge B \equiv D) \vee (A \equiv C \wedge B \to_K D) \vee (A \equiv D \wedge B \equiv C)$$

Further, by definition D 3 the followiny implications

$$(X \to_K Y) \Rightarrow X =_K Y, \quad X \equiv Y \Rightarrow X =_K Y$$

hold. Using them we immediately deduce

$$(7) \qquad (A*B \to_K C*D) \Rightarrow (A =_K C \wedge B =_K D) \vee (A =_K D \wedge B =_K C)$$

Consider now the case when $C*D$ is obtained from $A*B$ by two, three or more (but finite number) of uses of the law K, and suppose that[1]

$$(8) \quad (\exists U_1', \ldots, U_n')(\exists U_1'', \ldots, U_n'')\,[A*B \to_K U_1'*U_1'' \wedge \cdots \wedge U_n'*U_n'' \to_K C*D]$$

implies

$$(9) \qquad (A =_K C \wedge B =_K D) \vee (A =_K D \wedge B =_K D)$$

This is the induction hypothesis. Further, let $A*B$, $C*D$ be a pair of terms such that $C*D$ is obtained from $A*B$ by $n+1$ uses of the law K, i.e. there are terms $U_1', \ldots, U_{n+1}', U_1'', \ldots, U_{n+1}''$ such that

$$(10) \quad A*B \to_K U_1'*U_1'' \wedge \cdots \wedge U_n'*U_n'' \to_K U_{n+1}'*U_{n+1}'' \wedge U_{n+1}'*U_{n+1}'' \to_K C*D$$

Using the induction hypothesis from (10) we conclude

$$[(A =_K U_{n+1}' \wedge B =_K U_{n+1}'') \vee (A =_K U_{n+1}'' \wedge B =_K U_{n+1}')] \wedge (U_{n+1}'*U_{n+1}'' \to_K C*D)$$

Therefrom by using (7) we obtain immediately

$$[(A =_K U_{n+1}' \wedge B =_K U_{n+1}'') \vee (A =_K U_{n+1}'' \wedge B =_K U_{n+1}')] \wedge [(U_{n+1}' = C \wedge U_{n+1}'' = D) \vee (U_{n+1}' = D \wedge U_{n+1}'' = C)]$$

[1] As the law K is balanced, the term U_1 obtained from $A*B$ by one use of K mus be of the simmilar form, i.e. there are terms U_1' U_1'' such that $U_1 \equiv U_1'*U_1''$.

By the tautology $(p \vee q) \wedge (r \vee s) \Leftrightarrow (p \wedge r) \vee (p \wedge s) \vee (q \wedge r) \vee (q \wedge s)$ the preceding formula is equivalent to

$$
\begin{aligned}
&(A =_K U'_{n+1} \wedge B =_K U''_{n+1} \wedge U'_{n+1} =_K C \wedge U''_{n+1} =_K D) \\
&\vee (A =_K U'_{n+1} \wedge B =_K U''_{n+1} \wedge U''_{n+1} =_K D \wedge U''_{n+1} =_K C) \\
&\vee (A =_K U''_{n+1} \wedge B =_K U'_{n+1} \wedge U'_{n+1} =_K C \wedge U''_{n+1} =_K D) \\
&\vee (A =_K U''_{n+1} \wedge B =_K U'_{n+1} \wedge U'_{n+1} =_K D \wedge U''_{n+1} =_K C)
\end{aligned} \tag{11}
$$

Finaly, using the transitivity of $=_K$ and the compatibility of implication with conjunction from (11) we deduce (9) what completes the inductive proof.

The following lemma parallels to Lemma 1 for associative operations.

Lemma 2. *Let $*$ be a binary operation satisfying the equivalence* (3), *i.e.*

$$A*B = C*D \Leftrightarrow (A = C \wedge B = D) \vee (A = D \wedge B = C)$$

where A, B, C, D are any terms in $*$. Then $*$ is a commutative operation.

Proof. Namely, the implication

$$(A = C \wedge B = D) \vee (A = D \wedge B = C) \Rightarrow A*B = C*D \tag{12}$$

which immediately follows frow (3), may be replaced by two implications

$$A = C \wedge B = D \Rightarrow A*B = C*D, \quad A = D \wedge B = C \Rightarrow A*B = C*D$$

From the second of them taking $D \equiv A$, $C \equiv B$ we deduce

$$A = A \wedge B = B \Rightarrow A*B = B*A$$

therefrom it follows

$$A*B = B*A$$

Thus $*$ is a commutative operation.

Similar to the associative case the grouoid $(S, *)$ satisfying (12) need not to be a free commutative groupoid. This can be shown in a similar way by making the free commutative grorpoid generated by $\{a, b\}$ and introducing a new constant symbol c by definition Def(c).

3. In the following we give some applications of the equivalence (3)

(i) **The word problem.** We investigate whether for example the folloving words

$$((x*y)*x)*(y*(x*y)), \quad ((x*y)*x)*(x*(y*y)), \quad ((y*x)*y)*(x*(x*y))$$

are equivalent modulo commutative law, i.e. whether they are in the relation $=_K$. Using, the equivalence (3) several times we obtain for the first two words the following equivalence chain:

$$
\begin{aligned}
&((x*y)*x)*(y*(x*y)) = ((x*y)*x)*(x*(y*y)) \\
&\quad \Leftrightarrow [(x*y)*x = (x*y)*x \wedge y*(x*y) = x*(y*y)] \\
&\qquad \vee [(x*y)*x = x*(y*y) \wedge y*(x*y) = (x*y)*x[\\
&\quad \Leftrightarrow [((x*y = x*y \wedge x = x) \vee (x*y = x \wedge x = x*y)) \\
&\qquad \wedge ((y = x \wedge x*y = y*y) \vee (y = y*y \wedge x*y = x))] \\
&\qquad \vee [((x*y = x \wedge x = y*y) \vee (x*y = y*y \wedge x = x)) \\
&\qquad \wedge ((y = x*y \wedge x*y = x) \vee (y = x \wedge x*y = x*y))]
\end{aligned}
$$

Consider now each of the conjuctions:

$$x*y=x*y\wedge x=x,\ x*y=x\wedge x=x*y,\ y=x\wedge x*y=y*y,\ y=y*y\wedge x*y=x$$

$$x*y=x\wedge x=y*y,\ x*y=y*y\wedge x=x,\ y=x*y\wedge x*y=x,\ y=x\wedge x*y=x*y$$

As it holds:

$\tau(x*y=x*y)=\top$, $\tau(x=x)=\top$ — by reflexivity of $=_K$

$\tau(x*y=x)=\bot$, $\tau(x=x*y)=\bot$, $\tau(y=x)=\bot$, $\tau(x*y=y*y)=\bot$

$\tau(y=y*y)=\bot$, $\tau(x=y*y)=\bot$, $\tau(y=x*y)=\bot$ — as K is balanced

using these equalities we immediately conclude that the preceding conjunctions have the following truth values:

$$\top,\ \bot,\ \bot,\ \bot,\ \bot,\ \bot,\ \bot,\ \bot$$

Therefore the truth value of the last formula in the obtained equvalence chain is

$$[(\top\wedge\bot)\wedge(\bot\vee\bot)]\vee[(\bot\vee\bot)\wedge(\bot\vee\bot)],\ \text{i.e.}\ \bot$$

and the equality

$$((x*y)*x)*(y*(x*y))=((x*y)*x)*(x*(y*y))$$

does not hold.

For the first and third word we have the following equivalence chain:

$$((x*y)*x)*(y*(x*y))=((y*x)*y)*(x*(x*y))$$

$$\Leftrightarrow [(x*y=x\wedge x=x*y)\vee(x*y=x*y\wedge x=x)$$

$$\wedge[(y=y*x\wedge x*y=y)\vee(y=y\wedge x*y=y*x)$$

(After some computation)

$$\Leftrightarrow [\bot\wedge\top]\wedge[\bot\vee(\top\wedge x*y=y*x)]$$

(Since the formulae $x*y=x$, $y=y*x$, $x*y=y$ are false, while $x*y=x*y$, $x=x$, $y=y$ are true)

$$\Leftrightarrow x*y=y*x$$

$$\Leftrightarrow \top.$$

Thus, the first and third words are equivalent. As an immediate consequence of the preceding consideration and properties of the relation $=_K$ we obtain that the second and the third words are not equivalent.

It is not difficult to see that the given proofs could be shorter but the preceding ones have the following adwantage: they can be transformed into algorithm which is suitable for computer.[1] This fact follows from the next theorem.

[1] The decidability of the word-problem for free commutative groupoid can also be proved in some other ways—for example using the notion of context sensitive grammar, some general theorems about balanced laws etc.

Theorem 2. *Let A, B be terms built up from $*$, variables and constant symbols. Then there exist forumlae $F, G, \ldots, H$ such that the following equivalence*

$$A =_K B \Leftrightarrow F \vee G \vee \cdots \vee H \tag{13}$$

holds, where $F, G, \ldots, H$ are conjunctions of some formulae of the form

$$t =_K u,\ u =_K t,\ u =_K v,\ u =_K u \qquad (u, v \text{ are constant symbols or variables, } t \text{ is a term})$$

The equivalence (13) *can be obtained in finite number of steps using the equivalence* (3) *from left to right and the distibutive*[1] *laws*

$$(p_1 \vee p_2 \vee \cdots \vee p_r) \wedge (q_1 \vee q_2 \vee \cdots \vee q_s) \Leftrightarrow$$
$$\Leftrightarrow (p_1 \wedge q_1) \vee (p_1 \wedge q_2) \vee \cdots \vee (p_r \wedge q_s) \qquad (r, s = 1, 2, \ldots)$$

also from left to right.

Proof. We prove the theorem by induction on $o(B)$ — the number of the sings $*$ in B. If this number is 0 the assertion is true because in that case $A =_K B$ is of the form $A =_K u$, where u is a constant symbol or a variable. Suppose now that $o(B) > 0$ and that for all terms B such that $o(B) < n$ the assertion of theorem is true. Let further B be a term having the property $o(B) = n$. If $o(A) = 0$ the assertion is true and if not, there exist two terms A', A'' such that $A \equiv A' * A''$. To find A', A'' we need finite number of steps because the number of the subwords of the word A is finite. Similarly, as $o(B) = n$, $n > 0$, there exist terms B', B'' (which can be find in finite number of steps) such that $B \equiv B' * B''$. Thus we have equivalence

$$A =_K B \Leftrightarrow A' * A'' =_K A' * B'' \tag{14}$$

Using (4) we immediately obtain

$$A' * A'' =_K B' * B'' \Leftrightarrow (A' =_K B' \wedge A'' =_K B'') \vee (A' =_K B'' \wedge A'' =_K B') \tag{15}$$

As $o(B') < n$, $o(B'') < n$, by induction hypothesis it follows that there exist and can be found in finite number of steps some formulae

$$F_1, F_2, \ldots, F_k;\ G_1, G_2, \ldots, G_l;\ H_1, H_2, \ldots, H_m;\ I_1, I_2, \ldots, I_n$$

which are conjunction of the formulae having the form

$$t =_K u,\ u =_K t,\ u =_K v,\ u =_K u$$

such that the following equivalences

$$\begin{aligned} A' =_K B' &\Leftrightarrow F_1 \vee F_2 \vee \cdots \vee F_K \\ A'' =_K B'' &\Leftrightarrow G_1 \vee G_2 \vee \cdots \vee G_l \\ A' =_K B'' &\Leftrightarrow H_1 \vee H_2 \vee \cdots \vee H_m \\ A'' =_K B'' &\Leftrightarrow I_1 \vee I_2 \vee \cdots \vee I_n \end{aligned} \tag{16}$$

[1] In fact we apply also the associative laws for disjunction and conjunction but its application is not essential.

hold. Using the equivalences (13) — (16) we deduce

$$A =_K B \Leftrightarrow (F_1 \wedge G_1) \vee (F_1 \wedge G_2) \vee \cdots \vee (F_n \wedge G_l)$$
$$\vee (H_1 \wedge I_1) \vee (H_1 \wedge I_2) \vee \cdots \vee (H_m \wedge I_n)$$

As the formulae F_i, G_j, H_r, I_s are conjunctions of formulae of the form

(17) $t =_K u$, $u =_K t$, $u =_K v$, $u =_K u$ (u, v are constant symbols or variables, t is a term)

we conclude that the formulae

$$F_1 \wedge G_1,\ F_1 \wedge G_2,\ \ldots,\ F_k \wedge G_l,\ H_1 \wedge I_1,\ H_1 \wedge I_2,\ \ldots,\ H_m \wedge I_n$$

are of the same form. The proof of the theorem is completed.

Corollary. *Let A, B be terms is $*$. Then the formula $A = B$ is a consequence of the commutative law if and only if at least one of the formulae F, G, ..., H whose existance is proved in the preceding theorem is a conjunction of the formulae of the form $u =_K u$.*

Namely, it is not possible that any of the formulae of the form $u = t$, $t = u$, $u = v$, where u, v are diferent variables and t is a term, were a consequence of K because L is a balanced law. Thus, among the formulae (17) only $u =_K u$ can hold. Therefore at least one of the formulae F, G, ..., H is a consequence of K (this is a sufficient condition for $F \vee G \vee \cdots H$ to be a consequence of K) if and only if it is a conjunction of the form $u_1 =_K u_1 \wedge u_2 =_K u_2 \wedge \cdots \wedge u_n =_K u_n$.

We note that in the considered example we did not proceed in all details in the way deserибed in Theorem 2 because we used in the proof the facts like this: $x = x$ is a consequence of K, i.e. $x =_K x$ is true, $x * y =_K x$ is false etc.

The main reason for this is to obtain the proof not too long. We illustrate now in all details the proceeding described in Theorem 2. This is the proof for $(x*y)*x =_K x*(y*x)$:

$(x*y)*x =_K x*(y*x)$

$\Leftrightarrow [x*y =_K x \wedge x =_K y*x] \vee [x*y =_K y*x \wedge x =_K x]$

$\Leftrightarrow [x*y =_K x \wedge x =_K y*x]$

$\vee [((x =_K y \wedge y =_K x) \vee (x =_K x \wedge y =_K y)) \wedge x =_K x]$

$\Leftrightarrow [x*y =_K x \wedge x =_K y*x]$

$\vee [x =_K y \wedge y =_K x \wedge x =_K x]$

$\vee [x =_K x \wedge y =_K y \wedge x =_K x]$

Since the conjunction

$$x =_K x \wedge y =_K y \wedge x =_K x$$

is composed only of the formulae of the form $u =_K u$, the equality

$$(x*y)*x =_K x*(y*x)$$

holds.

(ii) **Solving of equations.** In this part of the paper we solve some equations on the free commutative groupoid. First of all the linear equation (in X):

(18) $$A*X = B$$

where A, B are given terms in $*$ and constant symbols (the names of elements of the generating set of the choosen free groupoid). Obviously, if B is without $*$ the equation (18) is impossible. Consider now the case B is of the form $B'*B''$. Using (4) we obtain

$$A*X = B'*B'' \Leftrightarrow (A = B' \wedge X = B'') \vee (A = B'' \wedge X = B')$$

wherefrom we conclude that the equation

$$A*X = B'*B''$$

is possible if and only if at least one of the conditions:

$$A = B', \quad A = B''$$

is satisfied. Further, we have:

(i) If $A = B'$, then the only solution is B''

(ii) If $A = B''$, then the only solution is B'

We solve now the equation (in $X_1, X_2, \ldots, X_n$, $n \geqslant 3$):

(19) $$((X_1*X_2)*X_3)\cdots*X_n = X_1*(X_2*\cdots(X_{n-1}*X_n)\cdots)$$

For example, if $n = 3$ the equation (19) reads:

$$(X_1*X_2)*X_3 = X_1*(X_2*X_3)$$

Using the equivalence (3) its solving is as follows:

$(X_1*X_2)*X_3 = X_1*(X_2*X_3)$

$\Leftrightarrow (X_1*X_2 = X_1 \wedge X_3 = X_2*X_3) \vee (X_1*X_2 = X_2*X_3 \wedge X_3 = X_1)$

$\Leftrightarrow X_1*X_2 = X_2*X_3 \wedge X_3 = X_1$.

(Since both equalities $X_1*X_2 = X_1$, $X_3 = X_2*X_3$ are impossible)

$\Leftrightarrow [(X_1 = X_2 \wedge X_2 = X_3) \vee (X_1 = X_3 \wedge X_2 = X_2)] \wedge X_3 = X_1$

$\Leftrightarrow [(X_1 = X_2 \wedge X_2 = X_3) \vee X_1 = X_3] \wedge X_1 = X_3$

(Since $X_2 = X_2$ is true)

$\Leftrightarrow X_1 = X_3$

(By the tautology $((p \vee q) \wedge q) \Leftrightarrow q$)

Thus we have proved the equivalence

(20) $$(X_1*X_2)*X_3 = X_1*(X_2*X_3) \Leftrightarrow X_1 = X_3$$

We prove now that, generally, the following equivalence (for $n \geqslant 3$)

$$(21) \qquad ((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_n = X_1 * (X_2 * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$
$$\Leftrightarrow X_1 = X_n \wedge X_2 = X_{n-1} \wedge \cdot \cdot \cdot \wedge X_{\left[\frac{n}{2}\right]} = X_{n-\left[\frac{n}{2}\right]+1}$$

holds, where $[x]$ is the whole number not exceeding x. The proof of (21) is for example:

$$(\cdot \cdot \cdot ((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-1}) * X_n = X_1 * (X_2 * \cdot \cdot \cdot (X_{n-2} * (X_{n-1} * X_n)) \cdot \cdot \cdot)$$
$$\Leftrightarrow [((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-1} = X_1 \wedge X_n = X_2 * (X_3 * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)]$$
$$\vee [((X_1 * X_2) * X_2) \cdot \cdot \cdot * X_{n-1} = X_2 * (X_3 * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot) \wedge X_n = X_1]$$
$$\Leftrightarrow ((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-1} = X_2 * (X_3 * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot) \wedge X_n = X_1$$

(Since both equalities in the first bracket are false)

We have proved the equivalence:

$$(\cdot \cdot \cdot ((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-1}) * X_n = X_1 * (X_2 * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$
$$\Leftrightarrow ((X_1 * X_2) * X_3) \cdot \cdot \cdot X_{n-1} = X_2 * (X_3 * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot) \wedge X_n = X_1$$

Let now the following equivalence

$$(\cdot \cdot \cdot ((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-1}) * X_n = X_1 * (X_2 * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$
$$(22) \qquad \Leftrightarrow ((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-i} = X_{i+1} * (X_{i+2} * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$
$$\wedge X_1 = X_n \wedge X_2 = X_{n-1} \wedge \cdot \cdot \cdot X_i = X_{n-i+1}$$

be the induction hypothesis. Applying (3) to the equality

$$((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-i} = X_{i+1} * (X_{i+2} * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$

we obtain

$$((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-1} = X_{i+1} * (X_{i+2} * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$
$$\Leftrightarrow [((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-i-1} = X_{i+1} \wedge X_{n-1} = X_{i+2} * (X_{i+3} * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$
$$\vee [((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-i-1} = X_{i+2} * (X_{i+3} * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot) \wedge X_{n-i} = X_{i+1}$$

Since the equality

$$((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-i-1} = X_{i+1}$$

is impossible in the case $i < \left[\frac{n}{2}\right]$, the preceding equivalence becomes

$$((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{n-i} = X_{i+1} * (X_{i+2} * \cdot \cdot \cdot * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot)$$
$$((X_1 * X_2) * X_3) \cdot \cdot \cdot X_{n-i-1} = X_{i+2} * (X_{i+3} * \cdot \cdot \cdot (X_{n-1} * X_n) \cdot \cdot \cdot) \wedge X_{n-i} = X_{i+1}$$

Substituing the right hand side of this equivalence into (22) we obtain:

$$(\cdots((X_1*X_2)*X_3)\cdots*X_{n-1})*X_n = X_1*(X_2*\cdots(X_{n-1}*X_n)\cdots)$$

$$\Leftrightarrow ((X_1*X_2)*X_3\cdots X_{n-i-1} = X_{i+2}*(X_{i+3}*\cdots(X_{n-1}*X_n)\cdots)$$

$$\wedge X_1 = X_n \wedge X_2 = X_{n-1} \wedge \cdots \wedge X_i = X_{n-i+1} \wedge X_{i+1} = X_{n-i}$$

where $i<\left[\frac{n}{2}\right]$. This completes the inductive proof of (22) in the case $1\leqslant i<\left[\frac{n}{2}\right]$. If $i=\left[\frac{n}{2}\right]$, the equivalence (22) becomes.

$$(\cdots((X_1*X_2)*X_3)\cdots*X_{n-1})*X_n = X_1*(X_2*\cdots(X_{n-1}*X_n)\cdots)$$

$$\Leftrightarrow ((X_1*X_2)*X_3)\cdots X_{n-\left[\frac{n}{2}\right]} = X_{\left[\frac{n}{2}\right]+1}*\left(X_{\left[\frac{n}{2}\right]+2}*\cdots(X_{n-1}*X_n)\cdots\right)$$

$$\wedge X_1 = X_n \wedge X_2 = X_{n-1} \wedge \cdots \wedge X_{\left[\frac{n}{2}\right]} = X_{n-\left[\frac{n}{2}\right]+1}$$

As the equality of the form

$$A = \tilde{A}$$

where $\tilde{A}$ is the mirror image of A is a consequence of K (what is a well known fact), the precending equivalence turns into (21). Namely if $n=2k$ then we have the fallowing equivalence chain

$$(\cdots((X_1*X_2)*X_3)\cdots*X_{2k-1})*X_{2k} = X_1*(X_2*\cdots(X_{2k-1}*X_{2k})\cdots)$$

$$\Leftrightarrow ((X_1*X_2)*X_3)\cdots*X_k = X_{k+1}*(X_{k+2}*\cdots(X_{2k-1}*X_{2k})\cdots)$$

$$\wedge X_1 = X_{2k} \wedge X_2 = X_{2k-1} \wedge \cdots \wedge X_k = X_{k+1}$$

$$\Leftrightarrow ((X_1*X_2)*X_3)\cdots*X_k = X_k*(X_{k-1}*\cdots*(X_2*X_1))$$

$$\wedge X_1 = X_{2k} \wedge X_2 = X_{2k-1} \wedge \cdots \wedge X_k = X_{k+1}$$

$$\Leftrightarrow X_1 = X_{2k} \wedge X_2 = X_{2k-1} \wedge \cdots \wedge X_k = X_{k+1}$$

While $(\cdots((X_1*X_2)*X_3)*X_4)\cdots)*X_k = X_k*(X_{k-1}*(\cdots(X_2*X_1)\cdots))$ is a consequence of K)

Thus, in the case $n=2k$ the following equivalence

$$(23) \qquad ((X_1*X_2)*X_3)\cdots*X_{2k} = X_1*(X_2*\cdots(X_{2k-1}*X_{2k})\cdots)$$

$$\Leftrightarrow X_1 = X_{2k} \wedge X_2 = X_{2k-1} \wedge \cdots \wedge X_k = X_{k+1}$$

holds.

In the case $n=2k+1$ we have the following equivalence chain

$$((X_1*X_2)*X_3)\cdots*X_{2k+1} = X_1*(X_2*\cdots(X_{2k}*X_{2k+1})\cdots)$$

$$\Leftrightarrow ((X_1*X_2)*X_3)\cdots*X_{k+1} = X_{k+1}*(X_{k+2}*\cdots(X_{2k}*X_{2k+1})\cdots)$$

$$\wedge X_1 = X_{2k+1} \wedge X_2 = X_{2k} \wedge \cdots \wedge X_k = X_{k+2}$$

$$\Leftrightarrow ((X_1 * X_2)X_3) \cdot \cdot \cdot * X_{k+1} = X_{k+1} * (X_k * \cdot \cdot \cdot (X_2 * X_1) \cdot \cdot \cdot)$$
$$\wedge X_1 = X_{2k+1} \wedge X_2 = X_{2k} \wedge \cdot \cdot \cdot \wedge X_k = X_{k+2}$$
$$\Leftrightarrow X_1 = X_{2k+1} \wedge X_2 = X_{2k} \wedge \cdot \cdot \cdot \wedge X_k = X_{k+2}$$

(For $(\cdot \cdot \cdot ((X_1 * X_2) * X_3) \cdot \cdot \cdot) * X_{k+1} = X_{k+1} * (X_k * \cdot \cdot \cdot (X_2 * X_1) \cdot \cdot \cdot)$ is a consequence of K)

Thus in the case $n = 2k+1$ we have proved the equivalence

(24) $$((X_1 * X_2) * X_3) \cdot \cdot \cdot * X_{2k+1} = X_1 * (X_2 * \cdot \cdot \cdot (X_{2k} * X_{2k+1}) \cdot \cdot \cdot)$$
$$\Leftrightarrow X_1 = X_{2k+1} \wedge X_2 = X_{2k} \wedge \cdot \cdot \cdot \wedge X_k = X_{k+2}$$

Written together (23) and (24) form the equivalence (21).

Using (21), i.e. (23) and (24) we immediately conclude that the solutions of the equation (19) are determined by

$$X_1 = \Pi_1, X_2 = \Pi_2, \ldots, X_k = \Pi_k, X_{k+1} = \Pi_k, X_{k+2} = \Pi_{k-1}, \ldots, X_{2k-1} = \Pi_2, X_{2k} = \Pi_1,$$

if $n = 2k$ i.e. by

$$X_1 = \Pi_1, \ X_2 = \Pi_2, \ldots, \ X_k = \Pi_k, \ X_{k+1} = \Pi_{k+1}, \ X_{k+2} = \Pi_k, \ X_{k+3} = \Pi_{k-1}, \ldots$$
$$\ldots, \ X_{2k} = \Pi_2, \ X_{2k+1} = \Pi_1, \text{ if } n = 2k+1,$$

where $\Pi_1, \Pi_2, \ldots, \Pi_k, \Pi_{k+1}$ are any elements of the considered free commutative groupoid

(iii) **Lawless operations.** By lawless operation we mean the operation holding no algebraic law except the law $x = x$. If a binary operation $*$ satisfres some algebraic laws Z it is possible that there exists an operation $\circ$ (defined by $*$) which is lawless. If this is the case and the laws Z are belanced then any Ω-algebra can be embedded in some groupoid of the variety Z. For example, any Ω-algebra can be embeded into a semigroup, since the operation $\circ$ defined by

$$x \circ y \overset{\text{def}}{=} (a * x) * y \qquad (a \text{ is a constant symbol})$$

is lawless, while $*$ satisfies asociative law.

Let now Z be a class of algebraic laws[1] having as consequences no laws of the form:

$u = v$ (u, v are different variables or constant symbols)

$t = u$ (u is a variable or constant, t is a term having at least one operation sign)

Further, let $\circ$ be a binary operation defined by

(25) $$x \circ y \overset{\text{def}}{=} \xi(x, y)$$

where $\xi(x, y)$ is a term in O built up from variables x, y and eventualy some new constant symbols. We now prove the following theorem which gives necessery and suficient conditions for $\circ$ to be a lawless operation.

[1] The laws Z are in the given operation language O.

Theorem 3. *Let Z be just descibed class of algebraic laws in the language O and $\circ$ the operation defined by* (25). *This operation is lawless, if and only if the fallowing equivalence*[1]

$$A\circ B=_Z C\circ D \Leftrightarrow A=_Z C\wedge B=_Z D \tag{26}$$

holds, where A, B, C, D are terms in $\circ$.

Proof. If $\circ$ is a lawless operation, then the equivalence

$$A\circ B=_Z C\circ D \Leftrightarrow A\equiv_\circ C\wedge B\equiv_\circ D$$

must be true, where $X\equiv_\circ Y$ means that the terms X, Y equals, as terms in $\circ$. From this the equivalence (26) follows immediately.

Suppose now that (26) holds and prove that $\circ$ is a lawlless operation, i.e. that

$$P=_Z Q \tag{27}$$

implies $P\equiv_\circ Q$, where P, Q are terms in $\circ$.

The proof is by induction in $o(Q)$ — the number of $\circ$ in Q. If $o(Q)=0$ and as the laws Z do not imply any equality of the from $t=u$ with $o(t)>0$, we conclude that the equility $o(P)=0$ holds. Thus, in the case $o(Q)=0$, the equality (27) is of the form $u=_Z v$, where u, v are variables or constant symbols. Because of the assumption that any equality of that form where u, v are different symbols does not hold, it follows that $u=_Z v$ implies $u\equiv v$.

Let now $o(Q)>0$ and suppose that for all terms Q with $o(Q)<n$ and for all P

$$P=_Z Q \Rightarrow P\equiv_\circ Q$$

holds. Further, let P, Q be terms in $\circ$ with $o(Q)=n$ $(n>0)$ and suppose $P=_Z Q$. If $o(P)=0$ then similarly as in the case $o(Q)=0$ we conclude $P\equiv_\circ Q$. Consider now the case $o(P)>0$. As $o(Q)>0$, the terms P, Q are of the form $P'\circ P''$, $Q'\circ Q''$, where P', P'', Q', Q'' are terms in $\circ$. Thus, the assumed equality reads

$$P'\circ P''=Q'\circ Q''$$

From this using (26) we deduce

$$P'=_Z Q',\ P''=_Z Q'' \tag{28}$$

As the terms Q', Q'' setisfy the conditions $o(Q')<n$, $o(Q'')<n$, using the induction hypothesis we obtain

$$P'\equiv_\circ Q',\ P''\equiv_\circ Q''$$

wherefrom it follows immeeiately

$$P'\circ P''\equiv_\circ Q'\circ Q'', \text{ i.e. } \quad P\equiv_\circ Q$$

which completes the induction proof. Thus the operation $\circ$ is lawless. We apply now the previous theorem to the commutative law. This law being balanced belongs to the laws for which the theorem holds. Thus, to prove that some

[1] The relation $=_Z$ is defined similarly as $=_K$, for example by definition: $A=_Z B$ iff $Z\vdash_E A=B$.

operation $\circ$ defined by $*$, where $*$ is the operation of the free commutative groupoid, is lawless it suffices to prove the equivalence (26). For example, each of the operations $\circ$ defined by

$$x \circ y \overset{\text{def}}{=} (x*x)*(x*y) \tag{29}$$

$$x \circ y \overset{\text{def}}{=} x*(x*y) \tag{30}$$

$$x \circ y \overset{\text{def}}{=} ((x*y)*x)*y \tag{31}$$

is lawless[1] Namely, if A, B, C, D are terms in $\circ$, where $\circ$ is defined by (29), then we have:

$A \circ B = C \circ D$

$\Leftrightarrow (A*A)*(A*B) = (C*C)*(C*D)$

$\Leftrightarrow A*A = C*C \wedge A*B = C*D$

(Using the equivalence (3)

$\Leftrightarrow [(A = C \wedge A = C) \vee (A = C \wedge A = C)] \wedge [(A = C \wedge B = D) \vee (A = D \wedge B = D)]$

$\Leftrightarrow A = C \wedge B = D$

(By tautology $p \vee (p \wedge q) \Leftrightarrow p$)

Thus, $A \circ B =_K C \circ D \Leftrightarrow A =_K C \wedge B =_K D$ wherefrom we couclude that the operation $\circ$ defined by (29) is lawless.

In the similar way it can be proved that the operations $\circ$ defined by (30) and (31) are lawless.

In what follows we prove that the operation $\circ$ defined by

$$x \bar{\circ} y = x*(y*y) \tag{32}$$

is lawless. The proof is based on the fallowing assertion:

If A is a term in $\bar{\circ}$, where $\circ$ is defined by (32) *then A cannot be expressed in the form $B*B$ where B is a term in $\circ$.*

which can easily be proved using the fact that each term in $\circ$ has even number of symbols $*$. Namely, if this term has one symbol $\circ$, then it is of the form $u*(v*v)$, where u, v are variables or constant symbols and it has two $*$'s. Further, if X and Y are terms in $\circ$ and if they have even number of $*$'s, so has the term $X*(Y*Y)$, i.e. $X \circ Y$.

Therefrom, we conclude: If A, B are terms in $\circ$ then A, $B*B$ have even and odd numer of $*$'s respectively and so the equality $A = B*B$ cannot hold[2]

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[1] We recall that the operation $\circ$ defined by (29) is that which was used in Kuratovski's definition of ordered pair by sets: $(x, y) = \{\{x, x\}, \{x, y\}\}$.

[2] Since applications of K cannot change the number of $*$'s.

Let, furter, A, B, C, D be terms in $\circ$, where $\circ$ is defined by (32). Then we have:

$A \circ B = C \circ D$

$\Leftrightarrow A*(B*B) = C*(D*D)$

$\Leftrightarrow (A = C \wedge B*B = D*D) \vee (A = D*D \wedge B*B = C)$

$\Leftrightarrow A = C \wedge B*B = D*D$

(Since both equalities $A = D*D$, $B*B = C$ are false)

$\Leftrightarrow A = C \wedge B = D$

Thus the operation $\circ$ defined by (32) is lawless.

Problem. Describe the classes of all semigroups and all commutative groupoids satisfying the equivalences (2) and (3) respectively.

For groupoids of the last of these classes all preceding considerations remain true.

REFFERENCES

[1] Cohn, P. M., *Universal algebra*, Harper & Row, New York, Evanston and London, 1965.

[2] Fraïssé, R., *Cours de logique mathématiques*, Gauthier-Villars, Paris, 1972.

[3] Lentin, A., *Contribution à une théorie des équations dans les monoides libres*, Thèse Fac. Sc. Paris, 1969.

[4] Lentin, A., *Equations dans les monoides libres*, Maths, et Sc. Humaines, *n°* 31 (1970), pp. 5—16.

[5] Levi, F. W., *On Semigroups*, Bull. Calcutta Math. Soc., 36, 1944, pp. 141—146.

[6] Prešić, M. D., Prešić, S. B. *On the embedding of* Ω*-algebras in groupoids*, Publ. Inst. Math. Beograd, t. 21 (35), 1977, pp. 169—174.

[7] Tarski, A., *Undecidability of the elementary theory of groups*, in *Tarski, A., Mostowski, A. and Robinson R. M.*, Undecidable Theories, North-Holland, Amsterdam, 1953.