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COMMON FIXED POINT OF CONTINUOUS MAPPINGS IN METRIC SPACES

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ABSTRACT. The paper deals with a class of selfmappings defined on a complete metric space. As a consequence it is shown that such a class of mappings has a unique fixed point under suitable restriction.

Recently, Singh [4] extended Jungck's result [2] and proved the following interesting result:

Theorem 1. *Let S and T be two continuous and commuting selfmappings of a complete metric space (X, d) satisfying the following conditions:*

(a) $S(X) \subset T(X)$;

(b) $$d(Sx, Sy) \leq ad(Tx, Ty) + b[d(Tx, Sx) + d(Ty, Sy)] + c[d(Tx, Sy) + d(Ty, Sx)]$$

for all x, y in X, where a, b and c are nonnegative real numbers satisfying

$$0 < a + 2b + 2c < 1.$$

Then S and T have a unique common fixed point.

Taking $b = c = 0$, we have Jungck's theorem [2].

The purpose of this paper is to generalize Singh's result by using another method. For related results, we refer to Jungck [1] and Raktoch [3].

Theorem 2. *Let E, F and T be three continuous selfmappings of a complete metric space (X, d) satisfying the following conditions:*

(C_1) $ET = TE$, $FT = TF$, $E(X) \subset T(X)$ *and* $F(X) \subset T(X)$;

(C_2) $$d(Ex, Fy) \leq a(d(Tx, Ty))\, d(Tx, Ty) + b(d(Tx, Ty))\,[d(Tx, Ex) + d(Ty, Fy)] + c(d(Tx, Ty))\,[d(Tx, Fy) + d(Ty, Ex)]$$

for all x, y in X, where a, b and c are monotonically decreasing functions from $R^+ \equiv [0, \infty)$ *into* $[0, 1)$ *satisfying*

$$a(t)+2b(t)+2c(t)<1$$

for all $t \in R^+$.

Then E, F and T have a unique common fixed point.

Proof. Let x_0 be a point in X. Let $x_1 \in X$ be such that $Tx_1 = Ex_0$ and let $x_2 \in X$ be such that $Tx_2 = Fx_1$. In general, we can chose x_{2n+1} and x_{2n+2} such that

$$Tx_{2n+1} = Ex_{2n},\ Tx_{2n+2} = Fx_{2n+1} \tag{1}$$

for $n=0, 1, \ldots$. We can do this since (C_1) holds. It follows from (C_2) that

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2n+2}) &= d(Ex_{2n}, Fx_{2n+1}) \\ &\le a(d(Tx_{2n}, Tx_{2n+1}))\, d(Tx_{2n}, Tx_{2n+1}) \\ &+ b(d(Tx_{2n}, Tx_{2n+1}))[d(Tx_{2n}, Ex_{2n}) + d(Tx_{2n+1}, Fx_{2n+1})] \\ &+ c(d(Tx_{2n}, Tx_{2n+1}))[d(Tx_{2n}, Fx_{2n+1}) + d(Tx_{2n+1}, Ex_{2n})] \end{aligned}$$

for $n=0, 1, 2, \ldots$. This implies

$$\begin{aligned} &d(Tx_{2n+1}, Tx_{2n+2}) \\ &\le \frac{a(d(Tx_{2n}, Tx_{2n+1})) + b(d(Tx_{2n}, Tx_{2n+1})) + c(d(Tx_{2n}, Tx_{2n+1}))}{1 - b(d(Tx_{2n}, Tx_{2n+1})) - c(d(Tx_{2n}, Tx_{2n+1}))}\, d(Tx_{2n}, Tx_{2n+1}) \\ &< d(Tx_{2n}, Tx_{2n+1}). \end{aligned}$$

Similarly, we can prove

$$d(Tx_{2n+2}, Tx_{2n+3}) < d(Tx_{2n+1}, Tx_{2n+2}).$$

Thus $\{d(Tx_n, Tx_{n+1})\}$ is decreasing. Let

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = k$$

and suppose that $k>0$. Put

$$\frac{a(k)+b(k)+c(k)}{1-b(k)-c(k)} = h.$$

Then $d(Tx_n, Tx_{n+1}) \ge k$ implies

$$\frac{a(d(Tx_n, Tx_{n+1})) + b(d(Tx_n, Tx_{n+1})) + c(d(Tx_n, Tx_{n+1}))}{1 - b(d(Tx_n, Tx_{n+1})) - c(d(Tx_n, Tx_{n+1}))} \le h$$

for $n=0, 1, 2, \ldots$. Thus

$$d(Tx_n, Tx_{n+1}) \le hd(Tx_{n-1}, Tx) \le \cdots \le h^n d(Tx_0, Tx_1)$$

Since $h \in (0, 1)$,

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{2}$$

Next, we show that $\{Tx_n\}_{n=0}^{\infty}$ is a Cauchy sequence. It is enough to show that $\{Tx_{2n}\}_{n=0}^{\infty}$ is a Cauchy sequence. Since

$$d(Tx_{2n}, Tx_{2m}) \leq d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2m})$$

and

$$\lim_{n\to\infty} d(Tx_{2n}, Tx_{2n+1}) = 0,$$

it is sufficient to show that $d(Tx_{2n+1}, Tx_{2m}) \to 0$ as $m, n \to \infty$. It follows from (C_2) that

$$\begin{aligned} d(Tx_{2n+1}, Tx_{2m}) &= d(Ex_{2n}, Fx_{2m-1}) \\ &\leq a(d(Tx_{2n}, Tx_{2m-1}))\, d(Tx_{2n}, Tx_{2m-1}) \\ &+ b(d(Tx_{2n}, Tx_{2m-1}))\, [d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2m-1}, Tx_{2m})] \\ &+ c(d(Tx_{2n}, Tx_{2m-1}))\, [d(Tx_{2n}, Tx_{2m}) + d(Tx_{2m-1}, Tx_{2n+1})] \end{aligned}$$

Thus

(3)
$$\begin{aligned} & d(Tx_{2n+1}, Tx_{2m}) \\ & \leq \frac{a(d(Tx_{2n}, Tx_{2m-1})) + b(d(Tx_{2n}, Tx_{2m-1})) + c(d(Tx_{2n}, Tx_{2m-1}))}{1 - a(d(Tx_{2n}\ Tx_{2m-1})) - 2c(d(Tx_{2n}, Tx_{2m-1}))} \\ & \times [d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2m-1}, Tx_{2m})]. \end{aligned}$$

For any $\varepsilon > 0$, there exists an N such that

(4)
$$\max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2m-1}, Tx_{2m})\} < \begin{cases} \dfrac{1}{2} \min\left\{\dfrac{\varepsilon[1 - a(\varepsilon) - 2c(\varepsilon)]}{a(\varepsilon) + b(\varepsilon) + c(\varepsilon)},\ 2\varepsilon\right\} & \text{if } a(\varepsilon) + b(\varepsilon) + c(\varepsilon) \neq 0; \\ \varepsilon & \text{if } a(\varepsilon) + b(\varepsilon) + c(\varepsilon) = 0 \end{cases}$$

for all $m, n \geq N$.

For those m, n such that $d(Tx_{2n}, Tx_{2m-1}) \geq \varepsilon$, we have, by (3) and (4),

$$d(Tx_{2n+1}, Tx_{2m}) \leq \frac{a(\varepsilon) + b(\varepsilon) + c(\varepsilon)}{1 - a(\varepsilon) - 2c(\varepsilon)} [d(Tx_{2n}, Tx_{2n+1}) + d(Tx_{2m-1}, Tx_{2m})] < \varepsilon,$$

since a, b and c are monotonically decreasing functions.

For those m, n such that $d(Tx_{2n}, Tx_{2m-1}) < \varepsilon$, we have, by (2),

$$d(Tx_{2n+1}, Tx_{2m}) \leq d(Tx_{2n+1}, Tx_{2n}) + d(Tx_{2n}, Tx_{2m-1}) + d(Tx_{2m-1}, Tx_{2m}) \leq \varepsilon$$

for m, n large enough.

Thus $\{Tx_n\}_{n=0}^{\infty}$ is a Cauchy sequence. By the completeness of X, $\{Tx_n\}_{n=0}^{\infty}$ converges to a point x in X. It follows from (1) that $\{Ex_{2n}\}_{n=0}^{\infty}$ and $\{Fx_{2n+1}\}_{n=0}^{\infty}$ also converge to x. Since E, F and T are continuous, we have

$$E(Tx_{2n}) \to Ex, \quad F(Tx_{2n+1}) \to Fx.$$

From (C_1)

$$E(Tx_{2n})=T(Ex_{2n}),\ F(Tx_{2n+1})=T(Fx_{2n+1})$$

for all $n=0, 1, 2, \ldots$. Taking $n\to\infty$, we have

$$Ex=Tx=Fx \tag{5}$$

and

$$T(Tx)=T(Ex)=E(Tx)=E(Ex)=T(Fx)=F(Tx)=F(Ex)=F(Fx) \tag{6}$$

By (C_2), (5) and (6),

$$\begin{aligned} d(Ex,\ F(Ex))&\le a(d(Tx,\ T(Ex))\,d(Tx,\ T(Ex)) \\ &+b(d(Tx,\ T(Ex))[d(Tx,\ Ex)+d(T(Ex),\ F(Ex))] \\ &+c(d(Tx,\ T(Ex))[d(Tx,\ F(Ex))+d(T(Ex),\ Ex)] \\ &=[a(d(Ex,\ F(Ex)))+2c(d(Ex,\ F(Ex)))]\,d(Ex,\ F(Ex)). \end{aligned}$$

Thus

$$[1-a(d(Ex,\ F(Ex))-2c(d(Ex,\ F(Ex))]\,d(Ex,\ F(Ex))\le 0.$$

This means

$$Ex=F(Ex). \tag{7}$$

It follows from (6) and (7) that Ex is a common fixed point of E, F and T.

Let u, v be two points of X such that $Eu=Fu=Tu=u$ and $Ev=Fv=Tv=v$. Then, by (C_2),

$$\begin{aligned} d(u,\ v)=d(Eu\ \ Fv)&\le a(d(Tu,\ Tv))\,d(Tu,\ Tv) \\ &+b(d(Tu,\ Tv))[d(Tu,\ Eu)+d(Tv,\ Fv)] \\ &+c(d(Tu,\ Tv))[d(Tu,\ Fv)+d(Tv,\ Eu)] \\ &=[a(d(u,\ v))+2c(d(u,\ v))]\,d(u,\ v). \end{aligned}$$

Hence $u=v$. Therefore our proof is complete.

Remark 1. Taking $E=F$, $b(t)=c(t)=0$ and a is a fixed real number in our Theorem 2, we get Jungck's theorem [2].

Remark 2. Let $a(t)$, $b(t)$ and $c(t)$ be fixed real numbers in Theorem 2. If $E=F$ then Singh's result [4] is a special case of our Theorem 2.

Remark 3. Let T be the identity mapping, $b(t)=c(t)=0$ and $E=F$, then Corollary of Rakotch [3] is a special case of our Theorem 2.

Corllary 1. *Let E, F and T be three continuous selfmappings of a complete metric space (X, d) satisfying (C_1) and the following condition*

(C_3) *there exist two positive integers m and n such that*

$$\begin{aligned} d(E^m x,\ F^n y)&\le a(d(Tx,\ Ty))\,d(Tx,\ Ty) \\ &+b(d(Tx,\ Ty))[d(Tx,\ E^m x)+d(Ty,\ F^n y)] \\ &+c(d(Tx,\ Ty))[d(Tx,\ F^n y)+d(Ty,\ E^m x)] \end{aligned}$$

for all x, y in X, where a, b and c are monotonically decreasing functions from R^+ into $[0, 1)$ satisfying

$$a(t)+2b(t)+2c(t)<1$$

Then E, F and T have a unique common fixed point.

Proof. It follows from (C_1) that $E^m T=TE^m$, $F^n T=TF^n$, $E^m(X)\subset E(X)\subset T(X)$ and $F^n(X)\subset F(X)\subset T(X)$. By Theorem 2, there exists a unique point x in X such that

$$x=Tx=E^m x=F^n x.$$

Also

$$T(Ex)=E(Tx)=Ex=E(E^m x)=E^m(Ex).$$

This means that Ex is a common fixed point of T and E^m. Similarly, Fx is a common fixed point of T and F^n. The uniqueness of x implies

$$Ex=Fx=Tx=x.$$

This completes our proof.

Corollary 2. *Let T and T_i $(i=1, 2, \dots, k)$ be continuous selfmappings of a complete metric space (X, d) satisfying the following conditions:*

(C_4) *$T_i T_j=T_j T_i$, $TT_i=T_i T$ for $i, j=1, 2, \dots, k$;*

(C_5) *$E(X)\subset T(X)$ where $E=T_1 T_2 \dots T_k$;*

(C_6) *for $E=F$, the condition (C_3) holds.*

Then T and T_i $(i=1, 2, \dots, k)$ have a common unique fixed point.

Proof. By Corollary 1, E and T have a unique common fixed point x in X. Thus $Ex=Tx=x$. Then

$$T_i(Ex)=T_i(Tx)=T_i x.$$

This together with the conditions (C_4) and (C_5) implies

$$E(T_i x)=T(T_i x)=T_i x.$$

Hence $T_i x$ $(i=1, 2, \dots, k)$ are common fixed points of E and T. By the uniqueness of the common fixed points of E and T, we have $T_i x=x$.

REFERENCES

[1] G. Jungck, *An iff fixed point criterion*, Math. Magazine, 49 (1976), 32—34.

[2] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, 83 (1976), 261—263.

[3] S. L. Singh, *On common fixed points of commuting mappings*, Math. Seminar Notes, 5 (1977), 132—134.

[4] E. Raktoch, *A note on contractive mappings*, Proc. Amer. Math. Soc., 13 (1962), 459—465.

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