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ON SOME NEW TREATEMENT OF ARTIFICIAL SATELLITE'S MOTION

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Absract

A few modification have been carried out with respect to the earlier solutions (Popović 1968, 1972) of the problem. A new anomaly, „regulating anomaly", was introduced. It enables to find for r the expression of the same form as for the unperturbed motion, while the „regulating anomaly" has only a small additional term. The expression for $s=\sin\varphi$ is in a closed form where from one easily gets the necessary approximations. Zero-, first and second approximations for r, s, and ζ are elaborated in detail. Solution for λ is the same as in the earlier papers since r, s, and ζ are indpendent of λ which can be calculated from sufficiently good approximations of r, s, and ζ.

Keywords: satellite's motion, regulating anomaly, approximation on ε^2.

Introduction

As in the two earlier papers (Popović 1968, 1972), the unknowns are

$$r, s=\sin\varphi,$$

(r, λ, φ are the spherical coordinates of satellite) while the equation

$$\zeta=\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)^2$$

is taken as solved for the successive approximation process:

$$\zeta\equiv\zeta_0+\varepsilon\zeta_1=\zeta_0-\varepsilon\int_{t_0}^{t}\frac{\partial U_1}{\partial s}\,ds \tag{0}$$

e. the following integrals are known

$$\left(\frac{d\vec{r}}{dt}\right)^2 = h + 2\frac{\mu}{r} - \varepsilon r^{-2} U_1(r, s),$$

(1)

$$U_1 = \frac{1}{r}(3s^2 - 1) + \varepsilon U_2 \qquad U_2 = 2\sum_{k=3}^{\infty} j_k r^{1-k} P_k(s)$$

(2)
$$\vec{k}\,\vec{r}\,\frac{d\vec{r}}{dt} = C$$

and, with the approximation of one degree higher than for r and s, ζ is known in the form (0).

The equations to be solved are

(3a)
$$\frac{d\lambda}{dt} = \frac{C}{r^2(1-s^2)}$$

which could be solved later (after r and s are found)

(3b)
$$\left(r^2\frac{ds}{dt}\right)^2 = \zeta(1-s^2) - C^2$$

$$\left(r\frac{dr}{dt}\right)^2 = r^2 h + 2\mu r - \zeta - \varepsilon U_1 \equiv$$

(3c)

$$\equiv r^2 h + 2\mu r - \zeta_0 - \varepsilon(\zeta_1 + U_1)\zeta_1 + U_1 = \int_{t_0}^{t} \frac{\partial U_1}{\partial r} dr.$$

The solution for r

One should handle first the equation (3c) since its main part depends only on r. On the basis of previous experience take

$$r = r_0(\rho + A\sin\xi - B\cos\xi)$$

(4)

$$dr = r_0[\rho' + (A' + B)\sin\xi - (B' - A)\cos\xi]\,d\xi$$

($'$ = derivative with respect to ξ), with ρ, A and B which satisfy

(5)
$$hr^2 + 2\mu r - \zeta_0 = -h\left(\frac{dr}{d\xi}\right)^2$$

that means (3c) to become

(6)
$$r\,d\xi = \sqrt{-h - \varepsilon(\zeta_1 + U_1)/\left(\frac{dr}{d\xi}\right)^2}\,dt.$$

This integral will not diverge when $dr/d\xi=0$ since $dr/dt=0$ in that case and, due to (3c), $\zeta_1+U_1=0$ when (5) holds for all ξ.

Conditions for (5) are obtained by equating the coefficients for $2\sin\xi\cos\xi$, $\cos^2\xi$, $2\sin\xi$, $2\cos\xi$ and the free terms in (5). By denoting

$$\theta^2=-\mu/(r_0 h)=a_0/r_0 \tag{7}$$

where $-\mu/a_0=h$ (for unperturbed motion) follows from (1), the conditions are

$$\begin{gathered}
-(A'+B)(B'-A)=AB\\
(B'-A)^2-(A'+B)^2=A^2-B^2\\
\rho'(A'+B)=\theta^2 A-\rho A\\
\rho'(B'-A)=\theta^2 B-\rho B\\
(\rho')^2+(A'+B)^2=\zeta_0/(r_0^2 h)+2\theta^2\rho-(\rho^2+A^2).
\end{gathered} \tag{8}$$

All these conditions could be satisfied although they are by 2 larger in number than the unkonwns (ρ, A, B). From the first two conditions one obtaines four solutions

$$B'-A=\pm A,\ A'+B=\mp B \tag{9}$$

$$B'-A=\pm B\sqrt{-1},\ A'+B=\pm A\sqrt{-2}. \tag{10}$$

The third and fourth conditions have the same value

$$\sqrt{-1}\,\rho'=\theta^2-\rho$$

using the solutions (10). The last condition (8) is then

$$\begin{gathered}
(\theta^2-\rho)^2+A^2=\zeta_0\,\theta^2/(\mu\, r_0)+(\rho-\theta^2)^2-\theta^4+A^2\\
\mu\, r_0\,\theta^2=\zeta_0,\ \text{resp.}\ \ \mu^2=-h\,\zeta_0.
\end{gathered}$$

Since $\zeta=(\vec{r}_0\times\vec{v}_0)^2$, the last condition can't be fulfilled already for the unperturbed motion, where $-h=\mu/a_0$.

Thus, only the solutions (9) are possible. The upper signs are not possible here since in the case of unperturbed motion ($A'=0$, $B'=0$) these equations give $A=0$, $B=0$ — which is impossible in the general case. Similary, $\rho'\neq 0$ is obtainable from the third and fourth conditions only if

$$\frac{A}{B}+\frac{B}{A}=0$$

which is also impossible in the general case. Therefore, one must have

$$\rho'=0,\ \rho=\theta^2=-\mu/(r_0 h). \tag{11}$$

Then (8) becomes

$$B^2 = -\zeta_0\, \theta^2/(\mu\, r_0) + \rho\theta^2 \left(2 + \rho \cdot \frac{hr_0}{\mu}\right) - A^2$$

$$A^2 + B^2 = \zeta_0/(hr_0^2) - \rho\mu/(hr_0) = (h\,\zeta_0 + \mu^2) : (h^2\, r_0^2).$$

This condition could be fulfilled for unperturbed motion. Hence it shows that in the perturbed motion the sum of squares of A and B must be a definitive constant

(12) $$A^2 + B^2 = H^2 = \frac{\mu^2 + h\,\zeta_0}{h^2\, r_0^2}.$$

The solutions for A and B from (9) should satisfy this condition, and one sees immediately that

(9a) $$BB' + AA' = 0$$

i. e. $A^2 + B^2 = \text{const.}$

Remark: Taking $\zeta_1 + U_1$ instead of ζ_0 in (5) could give H which is variable. However the fulfillment of the condition (9a) shows that H must be the constant and consequently it is impossible to include $\zeta_1 + U_1$ in (5).

By putting

(13) $$A = H \sin\gamma, \quad B = H\cos\gamma$$

the two equations (9) give pointly

(14) $$d\gamma = 0, \quad \gamma = \gamma_0.$$

The solutions (13) with the (constant value (14) taken together with (11) and (12) fulfill all the reguired conditions (8). Thus, (4) becomes

(15) $$r = a_0 + r_0\, H(\sin\gamma\sin\xi - \cos\gamma\cos\xi) = a_0 - r_0\, H\cos(\xi + \gamma)$$

and, by using (14),

(16) $$\frac{dr}{d\,\xi} = r_0\, H\sin(\xi + \gamma).$$

The calculation of the variable ξ

The new variable ξ could be found now from the equation (6) which now takes the form

(6a) $$[a_0 + r_0\, H(\sin\gamma\sin\xi - \cos\gamma\cos\xi)]\, d\,\xi = \sqrt{1 + \varepsilon\, M}\,\sqrt{-h}\, dt$$

(17) $$M = (\zeta_1 + U_1) : S, \quad S = hr_0^2\, H^2 \sin^2(\xi + \gamma).$$

One has, step by step, from (6a)

$$\left[\left(\frac{a_0}{r_0} - H\cos\gamma\right) + H\sin\gamma\sin\xi + H\cos\gamma\,(1-\cos\xi)\right]d\xi =$$

$$= \sqrt{1+\varepsilon M}\cdot\frac{1}{\theta r_0}\sqrt{\frac{\mu}{r_0}}\,dt$$

$$d\left[\left(\frac{a_0}{r_0} - H\cos\gamma\right)\xi + H\sin\gamma\,(1-\cos\xi) + H\cos\gamma\,(\xi-\sin\xi)\right] =$$

$$= (\sqrt{1+\varepsilon M} - 1 + 1)\,\frac{1}{\theta r_0}\sqrt{\frac{\mu}{r_0}}\,dt$$

The expression (15) gives for the initial moment ($\xi = 0$)

(18) $$r_0(1 + H\cos\gamma) = a_0, \text{ resp; } \left(\frac{a_0}{r_0} - H\cos\gamma\right) = 1.$$

If, in order to avoid the possibility of imaginary ξ, one takes

(19) $$\xi = y\theta,\ \frac{1}{a_0}\sqrt{\frac{\mu}{r_0}} = n,\ \alpha = \theta\cdot H\sin\gamma,\ \beta = \theta^2 H\cos\gamma$$

then one obtains

$$d(y + \alpha y^2 c_2 + \beta y^3 c_3) = \left(1 + \varepsilon\frac{M}{1+\sqrt{1+\varepsilon M}}\right)n\,dt$$

c_2, c_3 beeing two of always real functions:

(20) $$c_0 = \cos\xi,\ c_1 = \frac{\sin\xi}{\xi},\ c_2 = \frac{1-\cos\xi}{\xi^2},\ c_3 = \frac{\xi - \sin\xi}{\xi^3}.$$

The integration gives

$$y + \alpha y^2 c_2 + \beta y^3 c_3 = n(t - t_0) + \varepsilon n\int_0^{\varepsilon}\frac{M\,dt}{1+\sqrt{1+\varepsilon M}}$$

(21 $$y = \frac{n(t-t_0) + \varepsilon J}{1 + \alpha y c_2 + \beta y^3 c_3}$$

(22 $$J = \int_0^{\xi} n\frac{M\,dt}{1+\sqrt{1+\varepsilon M}}.$$

Finally (15) becomes

(15a) $$r = r_0(1 + \alpha y c_1 + \beta y^2 c_2).$$

The other variables

By making the transformations

(23) $$s = m \cdot \sin \eta, \quad m^2 = 1 - C^2/\zeta$$

as in the earlier papers (Popović 1968, 1972), and by using (6), (7) and then (14), (17), (15), the equation (3b) becomes

$$r \cdot \frac{ds}{d\xi} \cdot r \frac{d\xi}{dt} = \sqrt{\zeta}\, m \cos \eta$$

$$r (m \cos \eta\, d\eta + \sin \eta\, dm) \sqrt{-h(1+\varepsilon M)} = \sqrt{\zeta}\, m \cos \eta\, d\xi$$

$$d\eta = \sqrt{\frac{-\zeta}{h(1+\varepsilon M)}} \cdot \frac{d\xi}{a_0 - r_0 H \cos(\xi + \gamma)} - \frac{C^2\, d\zeta}{2\zeta^2 (1 - C^2/\zeta)} \operatorname{tg} \eta .$$

If one uses $\zeta = \zeta_0 + \varepsilon \zeta_1$ and the transformation

(24) $$\operatorname{tg} \frac{\xi + \gamma}{2} = \sqrt{\frac{1-e}{1+e}} \operatorname{tg} \frac{w}{2}, \quad e = r_0 H / a_0$$

(where e has the role of excentricity in the unperturbed motion) it has the form

$$d\eta = \sqrt{\frac{\zeta_0}{-h}} \sqrt{1 - \varepsilon \frac{M - \zeta_1/\zeta_0}{1 + \varepsilon M}} \frac{dw}{a_0 \sqrt{1 - e^2}} + \frac{\varepsilon}{2} \frac{C^2 \operatorname{tg} \eta}{\zeta(\zeta - C^2)} \frac{\partial U_1}{\partial s} ds$$

$$= \frac{1}{a_0} \sqrt{\frac{-\zeta_0}{h(1-e^2)}} \left[1 - \varepsilon \cdot \frac{(M - \zeta_1/\zeta_0) : (1 + \varepsilon M)}{1 + \sqrt{1 - \varepsilon (M - \zeta_1/\zeta_0) : (1 + \varepsilon M)}} \right] dw +$$

$$+ 3 \varepsilon \frac{C^2 \operatorname{tg}}{\zeta(\zeta - C^2)} \frac{s}{r} ds.$$

From (24) for e, (11) for H and $h = -\mu/a_0$, one gets

$$a_0^2 (1 - e^2) = a_0^2 - (\mu^2 + h \zeta_0) : h^2 = -\zeta_0 / h$$

$$\frac{1}{a_0^2} \frac{-\zeta_0}{h(1-e^2)} = \frac{-\zeta_0}{h(-\zeta_0/h)} = 1.$$

Without any restriction, it follows that

(25) $$\eta - \eta_0 = (w - w_0) + \varepsilon J_1$$

(26) $$J_1 = -\int_{w_0}^{w} \frac{M - \zeta_1/\zeta_0}{(1 + \varepsilon M) + \sqrt{(1 + \varepsilon M)\zeta/\zeta_0}} dw + $$
$$+ 3 C^2 \int_{w_0}^{w} \frac{\sin^2 \eta}{r \zeta^2} \left(d\eta + \operatorname{tg} \eta \cdot \frac{dm}{m} \right).$$

The procedure from an earlier paper (Popović 1972) is completely applicable here to find λ. It gives — the formula (45) in the original paper —

$$\lambda=\lambda_0+\operatorname{arc\,tg} T\Big|_{t_0}^{t}-3\,\varepsilon\, C\int_{t_0}^{t}\frac{1}{r\zeta\sqrt{\zeta}}\sin^2\eta\, d\eta-3\,\varepsilon\int_{t_0}^{t}\frac{T}{rm\zeta}\sin^2\eta\, dm-\tag{27}$$

$$-\frac{\varepsilon^2}{2}\int_{t_0}^{t}\frac{T}{m^2\zeta}\cdot\frac{\partial U_2}{\partial s}\, ds$$

$$T=\frac{C}{\sqrt{\zeta}}\operatorname{tg}\eta.\tag{28}$$

It seems that the somewhat simpler expression is obtained if in the expression for λ, having in mind that $s=\sin\varphi$, one introduced the transformation

$$C\operatorname{tg}\varphi=\sqrt{\zeta-C^2}\sin\theta$$

what gives

$$=C_2+\arcsin\left(\frac{C\operatorname{tg}\varphi}{\sqrt{\zeta-C^2}}\right)+\frac{Cm\operatorname{tg}\eta}{\zeta-C^2}\,d\zeta=$$

$$=C_2+\arcsin\left(\frac{C}{\sqrt{\zeta-C^2}}\cdot\frac{s}{\sqrt{1-s^2}}\right)+\frac{C}{2}\,\frac{m\operatorname{tg}\eta}{\zeta-C^2}\,d\zeta.$$

But the detailed transformation shows this is solution identical to (27).

To summarize the solution of the problem is given by the formulae

(0) for ζ and ζ_1, (1) for U_1 and U_2, (15) and (15a) for r,

with

$$\xi=y\theta,\quad \theta^2=\frac{a_0}{r_0}=-\frac{\mu}{r_0 h},\quad e=\frac{Hr_0}{a_0}$$

(20) for the functions c, (21) for y, with J from (22)

after wards (25) and (26) for $s=m\cdot\sin\eta$

and at the end

(27) and (28) for λ.

The first and second approximation for ζ

The zero-approximation is selfevident everywhere, although some expressions without ε give more than the zero-approximation. To obtain any further approximation one ought to find the corresponding parts with the previous approximation in the above expressions. In this process some difficulties with integration occur but at least all terms with ε could be integrated. The necessary transformations are performed in the sequel.

Firstly

$$m^2 = m_0{}^2 + C^2\left(\frac{1}{\zeta_0} - \frac{1}{\zeta}\right) = m_0{}^2 + \varepsilon\, C^2 \frac{\zeta_1}{\zeta_0\,\zeta} =$$

(29)

$$= m_0{}^2 + \varepsilon\left(\frac{C}{\zeta_0}\right)^2 \zeta_1 - \varepsilon^2\left(\frac{C\,\zeta_1}{\zeta_0}\right)^2$$

By utilizing (24) and $a_0{}^2(1-e^2) = -\zeta_0/h$, (15) becomes

(15b)

$$r = \frac{a_0(1-e^2)}{1+e\cos w} = \frac{\zeta_0}{\mu(1+e\cos w)}$$

With (1) and (25) the expression (0) gives

$$d\zeta_1 = -\frac{3}{r}\, d(s^2) - \varepsilon \frac{\partial U_2}{\partial s} ds = -\frac{3}{r}\sin 2\eta \cdot m^2\, d\eta -$$

$$-\frac{3}{r}\sin^2\eta\left(\frac{C}{\zeta}\right)^2 d\zeta - \varepsilon\frac{\partial U_2}{\partial s} ds.$$

The first item, with (29) and (15b) takes the form

$$-3\, m_0{}^2 \frac{\mu}{\zeta_0}(1+e\cos w)\sin 2\eta\, d\eta - 3\,\varepsilon\, C^2 \frac{\zeta_1}{\zeta_0\,\zeta}\,\frac{\sin 2\eta}{r}\, d\eta =$$

$$= \frac{3}{2} m_0{}^2 \frac{\mu}{\zeta_0}\, d(\cos 2\eta) - \frac{3}{2} m_0{}^2 \frac{\mu\, e}{\zeta_0}\,[\sin(2\eta + w) +$$

$$+ \sin(2\eta - w)]\, d(w + \varepsilon J_1) - 3\,\varepsilon\, C^2 \frac{\zeta_1 \sin 2\eta}{\zeta_0\,\zeta\, r}\, d\eta =$$

$$= \frac{3}{2} m_0{}^2 \frac{\mu}{\zeta_0}\, d(\cos 2\eta) + \frac{1}{2} m_0{}^2 \frac{\mu\, e}{\zeta_0}\,[d\cos(2\eta + w) + 3\, d\cos(2\eta - w) -$$

$$- \varepsilon\sin(2\eta + w)\, dJ_1 - 3\,\varepsilon\sin(2\eta - w)\, dJ_1] - 3\,\varepsilon\, C^2 \frac{\zeta_1 \sin 2\eta}{\zeta_0\,\zeta\, r}\, d\eta$$

and thus the whole expression is now

$$d\zeta_1 = m_0{}^2 \frac{\mu}{2\,\zeta_0}\, d\,[3\cos 2\eta + e\cos(2\eta + w) + 3\, e\cos(2\eta - w)] -$$

$$- \varepsilon\, m_0{}^2 \frac{\mu\, e}{\zeta_0}(2\sin 2\eta\cos w - \cos 2\eta\sin w)\, dJ_1 - 3\,\varepsilon\, C^2 \frac{\zeta_1 \sin 2\eta}{\zeta_0\,\zeta\, r}\, d\eta +$$

(30)

$$+ 3\,\varepsilon\, C^2 \frac{\sin^2\eta}{\zeta^2 r}\frac{\partial U_1}{\partial s}\, ds - \varepsilon\frac{\partial U_2}{\partial s}\, ds.$$

The terms in [ ] contain all that belongs to the first approximation for ζ. To obtain the second approximation one ought to take — termswith their zero-approximation. The details are worked out in the sequel. The integral of the main part, by using (25), is

$$\frac{m_0^2 \mu}{2\zeta_0}[3\cos 2(w+\eta_1)+e\cos(3w+2\eta_1)+3e\cos(w+2\eta_1)]_{w_0}^{w}$$

and the remainder can be written in the form

$$-m_0^2\frac{\mu}{2\zeta_0}[3\cos(2w+2\eta_1)J_{12}\varepsilon^2+3\sin(2w+$$

$$+2\eta_1)J_{11}\varepsilon+e\cos(3w+2\eta_1)J_{12}\varepsilon^2+e\sin(3w+2\eta_1)J_{11}\varepsilon+$$

$$+3e\cos(w+2\eta_1)J_{12}\varepsilon^2+3e\sin(w+2\eta_1)J_{11}\varepsilon]_{w_0}^{w}$$

where

(31) $$J_{11}=\frac{\sin(2\varepsilon J_1)}{\varepsilon},\qquad J_{12}=\frac{1-\cos(2\varepsilon J_1)}{\varepsilon^2}.$$

Therefore

$$\zeta_1=m_0^2\frac{\mu}{2\zeta_0}[3\cos(2w+2\eta_1)-3\cos 2\eta_0+e\cos(3w+2\eta_1)-$$

(32) $$-e\cos(w_0+2\eta_0)+3e\cos(w+2\eta_1)-3e\cos(2\eta_0-w_0)]+\varepsilon\zeta_2$$

$$\zeta_2=-m_0^2\frac{\mu}{2\zeta_0}\Big\{J_{11}[3\sin(2w+2\eta_1)+e\sin(3w+2\eta_1)]_{w_0}^{w}+$$

$$+\varepsilon J_{12}[3\cos(2w+2\eta_1)+e\cos(3w+2\eta_1)+3e\cos(w+2\eta_1)]_{w_0}^{w}$$

$$+2e\int_{w_0}^{w}(2\sin 2\eta\cos w-\cos 2\eta\sin w)\,dJ_1\Big\}-3C^2\int_{w_0}^{w}\frac{\zeta_1\sin 2\eta}{\zeta_0\zeta r}d\eta+$$

(33) $$+3C^2\int_{w_0}^{w}\frac{\sin^2\eta}{\zeta^2 r}\cdot\frac{\partial U_1}{\partial s}ds-\int\frac{\partial U_2}{\partial s}ds.$$

The first term of ζ_1 gives precisely the first approximation for $\zeta=\zeta_0+\varepsilon\zeta_1$. For the second approximation it is sufficient to take $J_{11}=2J_1(\varepsilon=0)$, to remove completely the term with J_{12}, and to take

$$\eta=w-w_0+\eta_0,\quad d\eta=dw$$

in the remainder.

The approximations for y

J from (22) is needed with necessary approximation to obtain the „regulating anomaly“ y from (21). In the same way, without any restrictions, by using (6) one has

$$dJ = n\frac{M}{1+\sqrt{1+\varepsilon M}}dt = \frac{n}{\sqrt{-h}}\frac{Mr}{2}\left[1+\frac{2-\sqrt{1+\varepsilon M}-1-\varepsilon M}{\sqrt{1+\varepsilon M}\,(1+\sqrt{1+\varepsilon M})}\right]d\xi$$

wherefrom

$$\frac{2\sqrt{-h}}{n}dJ = Mrd\xi - \varepsilon\frac{M^2(3-\varepsilon M)r\,d\xi}{2\sqrt{1+\varepsilon M}+(1+\varepsilon M)(2-\varepsilon M)}. \tag{34}$$

The first term is the most important. By using (17) for M, (15b) for r and the transformation (24) it follows

$$\frac{r}{S}d\xi = \frac{-\zeta_0(1+e\cos w)\left(1+\mathrm{tg}^2\frac{w}{2}\right)}{4a_0\mu^2e^2(1+e\cos w)\sqrt{1-e^2}\sin^2\frac{w}{2}} = \frac{-\sqrt{1-e^2}}{e^2\mu}\frac{dw}{\sin^2 w}. \tag{35}$$

To find ζ_1+U_1 one has

$$U_1 = (3m_0^2\sin^2\eta-1)\frac{\mu}{\zeta_0}(1+e\cos w)+\varepsilon\left(3C^2\frac{\zeta_1}{\zeta_0\zeta r}\sin^2\eta+U_2\right)=$$

$$=\frac{3m_0^2-2}{2\zeta_0}\mu(1+e\cos w)-3\frac{m_0^2\mu}{2\zeta_0}(1+e\cos w)[\cos(2w+2\eta_1)(1-\varepsilon^2J_{12})-$$

$$-\sin(2w+2\eta_1)\varepsilon J_{11}]+\varepsilon\left(3C^2\frac{\zeta_1\sin^2\eta}{\zeta_0\zeta r}+U_2\right) \tag{36}$$

$$\frac{2\zeta_0}{\mu}(\zeta_1+U_1) = (3m_0^2-2)(1+e\cos w)-3em_0^2\cos w\cdot\cos(2w+2\eta_1)-$$

$$m_0^2[3\cos 2\eta_0+e\cos(w_0+2\eta_0)+3e\cos(2\eta_0-w_0)]+em_0^2[\cos(3w+2\eta_1)+$$

$$+3e\cos(w+2\eta_1)]+\varepsilon\cdot\frac{2\zeta_0}{\mu}\left(\zeta_2-U_2+3C^2\frac{\zeta_1}{\zeta_0\zeta r}\sin^2\eta\right)+\varepsilon[J_{11}\sin(2w+2\eta_1)+$$

$$+\varepsilon J_{12}\cos(2w+2\eta_1)]\cdot 3m_0^2(1+e\cos w) \tag{36a}$$

$Mrd\xi = \frac{r}{S}d\xi\cdot(\zeta_1+U_1)$ from (35) and (36a).

The main part of this expression is easily integrated as

$$(J) = \frac{\zeta_0-C^2}{2e\zeta_0a_0\sqrt{r_0\zeta_0\mu}}\left\{\left(1-2\frac{C^2}{\zeta_0-C^2}\right)\frac{\cos w+e}{e\sin w}+\cos 2\eta_1\cdot\right.$$

$$\left.\cdot\frac{1}{\sin w}-2\sin(w+2\eta_1)\right\}_{w_0}^{w}-\left[\frac{3}{e}\cos 2\eta_0+4\cos w_0\cos 2\eta_0+\right.$$

$$\left.+2\sin w_0\sin 2\eta_0\right]\mathrm{ctg}\,w$$

and thus, with the ε-part of (34),

$$J=\left\{\left(1-2\frac{C^2}{\zeta_0-C^2}\right)\left(\frac{\cos w+e}{e\sin w}-\frac{\cos w_0+e}{e\sin w_0}\right)+\cos 2\eta_1\left(\frac{1}{\sin w}-\frac{1}{\sin w_0}\right)\right.$$

$$-2\sin(w+2\eta_1)+2\sin(2\eta_0-w_0)-\left[\left(\frac{3}{e}+4\cos w_0\right)\cos 2\eta_0+\right.$$

(37) $$\left.\left.+2\sin w_0\sin 2\eta_0\right](\operatorname{ctg} w-\operatorname{ctg} w_0)\right\}+\varepsilon J^+$$

$$J^+=\frac{-3(\xi_0-C^2)\sqrt{1-e^2}}{2e^2\zeta_0^2}\cdot\frac{n}{2\sqrt{-h}}\left\{\frac{2\zeta_0}{3\mu m_0^2}\int_{w_0}^{w}\left(\zeta_2+U_2+\right.\right.$$

$$\left.+3\frac{C^2\zeta_1\mu}{\zeta_0^2}\sin^2\eta\right)\frac{dw}{\sin^2 w}+\int_{w_0}^{w}[J_{11}\sin(2w+2\eta_1)+\varepsilon J_{12}\cos(2w+$$

(38) $$+2\eta_1)]\frac{dw}{\sin^2 w}-\frac{n}{2\sqrt{-h}}\int_{w_0}^{w}\frac{M^2(3-\varepsilon M)\,r\,d\xi}{2\sqrt{1+\varepsilon M}+(1+\varepsilon M)(2-\varepsilon M)}.$$

The approximations for S

J_1 from (26) has still to be elaborated. Denote the terms as $(J_1)_1$ and $(J_1)_2$ respectively. The first one could be rearranged as

(39) $$d(J_1)_1=-\frac{1}{2}\left(\frac{\zeta_1+U_1}{S}-\frac{\zeta_1}{\zeta_0}\right)(dw-\varepsilon\, dJ_{13})$$

(40) $$dJ_{13}=\frac{(3-\varepsilon M)M+(1+\varepsilon M)\xi_1/\zeta_0}{\sqrt{1+\varepsilon M}\,(\sqrt{1+\varepsilon M}+\sqrt{\zeta/\zeta_0})\,(1-\varepsilon M+\sqrt{(1+\varepsilon M)\zeta/\zeta_0})}\,dw.$$

Only the first part ought to be elaborated, since the zero-approximation suffices in the second one when η is supplied from (25) with the second approximation.

By utilizing (32) for ζ_1 and (36) for U_1, as well

(41) $$S=-\zeta_0 e^2\frac{\sin^2 w}{(1+e\cos w)^2}$$

from (17) and (24) — one has

$$\frac{\zeta_1+U_1}{S}-\frac{\zeta_1}{\zeta_0}=-\left[\frac{(1+e\cos w)^2}{\zeta_0 e^2\sin^2 w}+\frac{1}{\zeta_0}\right]\left\{\frac{m_0^2\mu}{2\zeta_0}[\cos(2w+2\eta_1)(3+4e\cos w)+\right.$$

$$\left.+2e\sin w\sin(2w+2\eta_1)-(3+4e\cos w_0)\cos 2\eta_0-2e\sin w_0\sin 2\eta_0]+\varepsilon\zeta_2\right\}-$$

$$-\frac{(1+e\cos w)^2}{\zeta_0 e^2 \sin^2 w}\left\{\frac{\mu}{2\zeta_0}(1+e\cos w)\,[3\,m_0^2-2-3\,m_0^2\cos(2\,w+\right.$$

$$+2\,\eta_1)\,(1-\varepsilon^2 J_{12})+3\,m_0^2\sin(2\,w+2\,\eta_1)\,\varepsilon J_{11}+$$

$$\left.+\varepsilon\left(3\,\frac{C^2\zeta_1\mu}{\zeta_0^2}\sin^2\eta+U_2\right)\right\}.$$

Finally, after the integration, one has

$$(J_1)_1=\frac{m_0^2\,\mu}{4\,e^2\,\zeta_0^2}\int\limits_{w_0}^{w}\frac{1+e^2+2\,e\cos w}{\sin^2 w}\,[\cos 2\,\eta_1\cdot e\,(\cos w+2\sin^2 w\cos w)+$$

$$+2\,e\sin 2\,\eta_1\sin w\,(1-2\sin^2 w-\cos^2 w)+(3-2\,m_0^{-2})\,(1+e\cos w)-$$

$$-(3+4\,e\cos w_0)\cos 2\,\eta_0-2\,e\sin w_0\sin 2\,\eta_0]\,dw-$$

$$-\frac{\mu\,m_0^2}{4\,\zeta_0^2}\int\limits_{w_0}^{w}(1+e\cos w)\,[3-2\,m_0^{-2}-3\cos(2\,w+2\,\eta_1)]\,dw+\varepsilon\,(J_1)_1^{+}$$

$$(J_1)_1=\frac{m_0^2\,\mu}{4\,e^2\,\zeta_0^2}\left\{2\,e\,(1+e^2+e^2\cos^2 w)\sin(w+2\,\eta_1)+\frac{5}{2}\,e^2\sin(2\,w+2\,\eta_1)-\right.$$

$$-e\,\frac{(1+e\cos w)^2}{\sin w}(\cos 2\,\eta_1+3)-3\,e^2 w\,(3-2\,m_0^{-2})+$$

$$+2\,m_0^{-2}\left[(1+3\,e^2)\,\mathrm{ctg}\,w+e\cdot\frac{3+e^2}{\sin w}+e^3\sin w\right]+2\,e\,[\cos(2\,\eta_0-w_0)+$$

$$\left.+\cos w_0\cos 2\,\eta_0]\cdot\frac{\cos w+e+e\,(1+e\cos w)}{\sin w}\right\}_{w_0}^{w}+\varepsilon\,(J_1)_1^{+}$$

(42)

$$(J_1)_1^{+}=\frac{1}{2}\int\limits_{w_0}^{w}\frac{1+e^2+2\,e\cos w}{\zeta_0\,e^2\sin^2 w}\,\zeta_2\,dw+\frac{1}{2}\int\limits_{w_0}^{w}\frac{(1+e\cos w)^2}{\zeta_0\,e^2\sin^2 w}\times$$

$$\times\frac{3\,m_0^2\,\mu}{2\,\zeta_0}(1+e\cos w)\left[J_{11}\sin(2\,w+2\,\eta_1)-\varepsilon\,J_{12}\cos(2w+2\,\eta_1)+\right.$$

$$\left.+3\,\frac{C^2\zeta_1\mu}{\zeta_0^2}\sin^2\eta+U_2\right\}dw+\frac{1}{2\,\zeta_0}\int\limits_{w_0}^{w}\left[\left(\frac{1+e\cos w}{e\sin w}\right)^2(\zeta_1+U_1)+\zeta_1\right]dJ_{13}. \tag{43}$$

In a similar way, by using (25), (23) and (0), one has

$$d\,(J_1)_2=3\,C^2\,\frac{\sin^2\eta}{\zeta^2 r}\left(dw+\varepsilon\,dJ_1-\varepsilon\,\mathrm{tg}\,\frac{C^2}{rm^2\zeta^2}\,\frac{\partial U_1}{\partial s}\,ds\right)$$

where the main part, by utilizing (15b) and (25), can be written in the form

$$\frac{1}{2}C^2\mu\zeta_0^{-3}d\left[3w+3e\sin w-\frac{3}{2}\sin 2\eta-e\sin(2\eta+w)-3e\sin(2\eta-w)\right]+$$

$$+\frac{\varepsilon}{2}C^2\mu\zeta_0^{-3}[3\cos 2\eta+2e\cos(2\eta+w)+6e\cos(2\eta-w)]\,dJ_1.$$

If one takes the transformations

$$\sin 2\eta=\sin(2w+2\eta_1)-\sin(2w+2\eta_1)\varepsilon^2 J_{12}+\cos(2w+2\eta_1)\varepsilon J_{11}$$

$$\sin(2\eta+w)=\sin(3w+2\eta_1)-\sin(3w+2\eta_1)\varepsilon^2 J_{12}+\cos(3w+2\eta_1)\varepsilon J_{11}$$

$$\sin(2\eta-w)=\sin(w+2\eta_1)-\sin(w+2\eta_1)\varepsilon^2 J_{12}+\cos(w+2\eta_1)\varepsilon J_{11}$$

the integration gives

$$(J_1)_2=\frac{1}{2}C^2\mu\zeta_0^{-3}\Big[3w+3e\sin w-\frac{3}{2}\sin(2w+2\eta_1)-e\sin(3w+2\eta_1)-$$

(44) $$-3e\sin(w+2\eta_1)\Big]_{w_0}^{w}+\varepsilon(J_1)_2^{+}$$

$$(J_1)_2^{+}=\frac{1}{2}C^2\mu\zeta_0^{-3}\Big[-\frac{3}{2}J_{11}\cos(2w+2\eta_1)+\frac{3}{2}\varepsilon J_{12}\sin(2w+2\eta_1)-$$

$$-eJ_{11}\cos(3w+2\eta_1)+e\varepsilon J_{12}\sin(3w+2\eta_1)-$$

$$-3eJ_{11}\cos(w+2\eta_1)+3eJ_{12}\sin(w+2\eta_1)\Big]+$$

$$+\frac{1}{2}C^2\mu\zeta_0^{-3}\int_{w_0}^{w}|3\cos 2\eta+2e\cos(2\eta+w)+6e\cos(2\eta-w)|\,dJ_1+$$

(45) $$+3C^2\int_{w_0}^{w}\frac{\sin^2\eta}{\zeta^2 r}\left[dJ_1-\mathrm{tg}\,\eta\cdot\frac{C^2}{2m^2\zeta^2}\frac{\partial U_1}{\partial s}ds\right].$$

Finally, by combining everything,

$$J_1=(J_1)_1+(J_1)_2 \text{ from (42) and (44).}$$

Evaluating w and λ

Many of those expressions depend on the wariable w and it **has to be** expressed as a function of the anomaly y by connecting (15a) and (15b) for r, nemely

$$r_0(1+\alpha y c_1+\beta y^2 c_2)=\frac{\zeta_0}{\mu(1+e\cos w)}$$

i. e.

(46) $$e\cos w = \frac{\zeta_0}{r_0\,\mu\,(1+\alpha\,y c_1+\beta\,y^2 c_2} - 1.$$

Finally, the variable λ can be evaluated from (27) and (28) as soon as the other variables are obtained with the corresponding approximation.

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