

NOTES ON SOME INEQUALITIES OF P. M. VASIĆ

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1. A function $f:[a, b] \rightarrow R$ is called n -convex on $[a, b]$ if and only if, for every $x_0, x_1, \dots, x_n \in [a, b]$

$$[x_0, \dots, x_n; f] \geq 0,$$

where $[x_0, \dots, x_n; f]$ is defined by the recursive relation

$$[x_0, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0}, \quad [x; f] = f(x).$$

Let

$$G_k(x) = f(x) - \sum_{j=0}^{k-1} \frac{f^{(j)}(c)}{j!} (x-c)^j, \quad \text{for } k=1, \dots, n-1.$$

G. V. MILOVANOVIĆ [1] proved:

Theorem 1. Let f be a n -convex function on $[a, b]$ ($n \geq 2$). Then for every $c \in [a, b]$, $x \mapsto G_{n-1}(x)/(x-c)^{n-1}$ is a nondecreasing function on $[a, b]$.

We shall prove the following result:

Theorem 2. Let f be a n -convex function on $[a, b]$. Then for every $c \in [a, b]$ it holds that

$$F_k(x) = \begin{cases} \frac{G_k(x)}{(x-c)^k}, & \text{for } x \neq c, \\ \frac{1}{k!} f^{(k)}(c), & \text{for } x = c, \end{cases}$$

is a $(n-k)$ -convex function on $[a, b]$ ($k=1, \dots, n-2$).

Proof. Assuming the following notation

$$[x_0, \dots, x_n; f] \equiv [x_0, \dots, x_n; f(x)].$$

We shall prove, now, the following identity

$$(1) \quad [x_0, \dots, x_n; f(x)] = \left[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n; \frac{f(x) - f(x_i)}{x - x_i} \right],$$

$$(i=0, \dots, n).$$

For $n=1$, (1) is true. Suppose that (1) is valid for n . Then

$$\begin{aligned} [x_0, \dots, x_{n+1}; f(x)] &= \frac{[x_1, \dots, x_{n+1}; f(x)] - [x_0, \dots, x_n; f(x)]}{x_{n+1} - x_0} \\ &= \frac{\left[x_1, \dots, x_n; \frac{f(x) - f(x_{n+1})}{x - x_{n+1}} \right] - \left[x_1, \dots, x_n; \frac{f(x) - f(x_0)}{x - x_0} \right]}{x_{n+1} - x_0} \\ &= \left[x_1, \dots, x_n; \frac{\frac{f(x) - f(x_{n+1})}{x - x_{n+1}} - \frac{f(x) - f(x_0)}{x - x_0}}{x_{n+1} - x_0} \right] \\ &= \left[x_1, \dots, x_n; \frac{\frac{f(x) - f(x_0)}{x - x_0} - \frac{f(x_{n+1}) - f(x_0)}{x_{n+1} - x_0}}{x - x_{n+1}} \right] \\ &= \left[x_1, \dots, x_n, x_{n+1}; \frac{f(x) - f(x_0)}{x - x_0} \right]. \end{aligned}$$

By analogy, we can prove that (1) is also valid for $i=1, \dots, n$.

According to identity (1), it is obviously, that Theorem 1, for $k=1$, is true. Suppose that it is true for any k . Then

$$F(x) = \begin{cases} \frac{\frac{G_k(x)}{(x-c)^k} - \frac{1}{k!} f^{(k)}(c)}{x-c} = \frac{G_{k+1}(x)}{(x-c)^{k+1}}, & \text{for } x \neq c, \\ \lim_{x \rightarrow c} \frac{d}{dx} \frac{G_k(x)}{(x-c)^k} = \frac{1}{(k+1)!} f^{(k+1)}(c), & \text{for } x=c, \end{cases}$$

is a $(n-k-1)$ -convex function. Thus, the theorem is proved.

Remarks: 1° From Theorem 2, for $k=n-2$, we can easily get Theorem 1.

2° In Theorems 1 and 2 we suppose that $f^{(i)}(a)=f^{(i)}(a+)$ and $f^{(i)}(b)=f^{(i)}(b-)$ ($i=1, 2, \dots, n-2$).

2. P. M. VASIĆ [2] proved:

Theorem 3. Let f be a real 3-convex function on $[0, a]$, $a_i \in [0, a]$ for $i = 1, \dots, n$ and $\sum_{i=1}^n a_i \in [0, a]$. Then

$$(2) \quad \begin{aligned} & \sum_{1 \leq i < j \leq n} f(a_i + a_j) + \binom{n-1}{2} f(0) \leq \\ & \leq (n-2) \sum_{i=1}^n f(a_i) + f\left(\sum_{i=1}^n a_i\right), \quad (n \geq 3). \end{aligned}$$

Remark: 3° Further generalizations of Theorem 3 gave also in [3], [4] and [5].

Now, we shall give a new proof of Theorem 3. From (2), we get

$$\begin{aligned} \text{i. e. } & \sum_{1 \leq i < j \leq n} (f(a_i + a_j) - f(0)) \leq (n-2) \sum_{i=1}^n (f(a_i)) - f(0) + \left(f\left(\sum_{i=1}^n a_i\right) - f(0) \right), \\ & \sum_{1 \leq i < j \leq n} (a_i + a_j) F(a_i + a_j) \leq (n-2) \sum_{i=1}^n a_i F(a_i) + \sum_{i=1}^n a_i F\left(\sum_{j=1}^n a_j\right), \\ \text{i. e. } & (3) \quad \sum_{i=1}^n a_i \left((n-2) F(a_i) - \sum_{\substack{j=1 \\ j \neq i}}^n F(a_i + a_j) + F\left(\sum_{j=1}^n a_j\right) \right) \geq 0, \end{aligned}$$

where $F(x) = \frac{f(x) - f(0)}{x}$. Function F , according to Theorem 2, is convex.

The inequality (3) holds if

$$(4) \quad F\left(\sum_{i=1}^n a_i\right) + (n-2) F(a_i) \geq \sum_{\substack{j=1 \\ j \neq i}}^n F(a_i + a_j), \quad (i = 1, \dots, n).$$

Without losing a generality, we can suppose that $a_1 \geq \dots \geq a_n$. Denote

$$\begin{aligned} x_1 &= \sum_{i=1}^n a_i, \quad x_j = a_i \quad (j = 2, \dots, n-1); \quad y_j = a_i + a_j \quad (j = 1, \dots, i-1), \\ & y_{j-1} = a_i + a_j \quad (j = i+1, \dots, n). \end{aligned}$$

Then, by using well-known Majorization theorem (see [6, p. 64]), we obtain (4). Thus, the theorem is proved.

3. P. M. VASIĆ [7] has given the following generalization of the inequality by D. MARKOVIĆ [8]:

Theorem 4. Let function f has $k+1$ derivatives on $I = [0, a]$ and let f be $(k+1)$ -convex on I . Suppose that $f^{(m)}(0) = 0$ ($m = 1, \dots, k-1$). Then

$$(5) \quad f\left(\left(\frac{1}{n} \sum_{i=1}^n x_i^k\right)^{\frac{1}{k}}\right) \leq \frac{1}{n} \sum_{i=1}^n f(x_i) \leq \frac{1}{n} \left(f\left(\left(\sum_{i=1}^n x_i^k\right)^{\frac{1}{k}}\right) + (n-1) f(0)\right)$$

if

$$x_i \in I \quad (i = 1, \dots, n), \quad \left(\sum_{i=1}^n x_i^k\right)^{1/k} \in I.$$

G. V. MILOVANOVIC and J. E. PEČARIĆ [9] proved:

Theorem 5. Let the integrable functions f and g satisfy the conditions:

- 1° f is k -convex ($k \in N$),
- 2° $f^{(m)}(a) = 0$ ($m = 0, 1, \dots, k-2$),
- 3° $0 \leq g(x) \leq 1$ ($\forall x \in [a, b]$).

Then

$$(6) \quad \int_a^{a+\lambda} f(x) dx \leq \int_a^b f(x) g(x) dx,$$

where

$$\lambda = \left(k \int_a^b (x-a)^{k-1} g(x) dx \right)^{1/k}.$$

By using Theorem 5 we can prove Theorem 4, which is valid for certain less strong conditions, i. e. the following theorem holds:

Theorem 6. Let $f: [a, b] \rightarrow R$, $a_i \in [a, b]$ and p_i ($i = 1, \dots, n$) satisfy the conditions:

- 1° f is $(k+1)$ -convex,
- 2° $f^{(m)}(a) = 0$ ($m = 1, \dots, k-1$),
- 3° $a = (a_1, \dots, a_n)$ is a monotone sequence,
- 4° $0 \leq \sum_{i=1}^j p_i \leq \sum_{i=1}^n p_i$ ($j = 1, \dots, n-1$), $\sum_{i=1}^n p_i > 0$.

Then the following inequality holds

$$f\left(a + \left(\frac{\sum_{i=1}^n p_i (a_i - a)^k}{\sum_{i=1}^n p_i}\right)^{\frac{1}{k}}\right) \leq \frac{\sum_{i=1}^n p_i f(a_i)}{\sum_{i=1}^n p_i}$$

Proof. Let $a \leq a_1 \leq \dots \leq a_n$ and

$$(7) \quad g(t) = g_i \text{ for } a_{i-1} < t \leq a_i \quad (a_0 = a); \quad g_i = \frac{\sum_{j=1}^i p_j}{\sum_{j=1}^n p_j}.$$

Function $f'(x)$ satisfies the conditions of Theorem 5. So,

$$\begin{aligned} \lambda &= \left(k \int_a^{a_n} (x-a)^{k-1} g(x) dx \right)^{\frac{1}{k}} = \left(\sum_{i=1}^k g_i ((a_i - a)^k - (a_{i-1} - a)^k) \right)^{\frac{1}{k}} \\ &= \left(\frac{\sum_{i=1}^n p_i (a_i - a)^k}{\sum_{i=1}^n p_i} \right)^{\frac{1}{k}} \end{aligned}$$

and

$$\int_a^{a+\lambda} f'(t) dt \leq \int_a^{a_n} f'(t) g(t) dt = \sum_{j=1}^n (f(a_j) - f(a_{j-1})) g_j$$

i. e.

$$f(a+\lambda) \leq \frac{\sum_{i=1}^n p_i f(a_i)}{\sum_{i=1}^n p_i}.$$

By analogy we get (7) for $a \leq a_n \leq \dots \leq a_1$. Thus the theorem is proved.

By substitution $(a_i - a)^k = t_i$ ($i = 1, \dots, n$), from Theorem 6 we get that $t \mapsto f(a + t^{1/k})$ is a convex function on $[0, (b-a)^k]$. Now, we can prove the following generalization of Theorem 6:

Theorem 7. Let the conditions 3° and 4° of Theorem 6, hold. If $t \mapsto f(a + t^{1/k})$ is a convex function on $[0, (b-a)^k]$, then (7) holds.

Proof. By using JENSEN-STEFFENSEN inequality we get

$$f\left(a + \left(\frac{\sum_{i=1}^n p_i t_i}{\sum_{i=1}^n p_i}\right)^{\frac{1}{k}}\right) \leq \frac{\sum_{i=1}^n p_i f(a + t_i^{1/k})}{\sum_{i=1}^n p_i}.$$

By substitution $t_i = (a_i - a)^k$ ($i = 1, \dots, n$), we obtain (7).

Corollary 8. Let f is two times differentiable function on $[a, b]$ such that

$$(x-a) f''(x) - (k-1) f'(x) \geq 0 \quad (\forall x \in [a, b]),$$

and let $k \in N$. Then (7) holds.

Remarks: 4° Corollary 8, for $a=0$, is proved in [10]. In [10] the other interesting generalizations of inequality (5) are given.

5° Applying the method which is given in proof of Theorem 7 it is possible to find corresponding analogies for a lot of inequalities valid for convex functions. For example, from well-known PETROVIĆ's inequality we get the second inequality of (5).

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