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ON A PAPER OF SAHA AND RAY

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1. *Introduction.* In this note several basic mistakes in the paper of Saha and Ray [4] are corrected.

2. *Results.* Let R^1 denote the set of all real numbers and $\mathcal{L}^1$ the family of all Lebesgue measurable subsets of R^1. If $A \in \mathcal{L}^1$ then $|A|$ will stand for the Lebesgue measure of the set A. Suppose that to each element ω belonging to a metric space Ω a certain transformation T_ω of $\mathcal{L}^1$ into $\mathcal{L}^1$ is associated. Neubrunn and Šalát [2] considered such families of transformations satisfying the following three conditions.

(a) There exists $\omega_0 \in \Omega$ such that for every closed interval $\langle a, b \rangle$ and every sequence $\{\omega_n\}_{n=1}^{\omega}$ of elements belonging to Ω and converging to ω_0,

$$\lim_{n\to\infty} (\inf T_{\omega n}(\langle a, b\rangle)) = a, \ \lim_{n\to\infty} (\sup T_{\omega n}(\langle a, b\rangle)) = b \text{ holds;}$$

(b) If $E, F \in \mathcal{L}^2$ and $E \subset F$ then for every $\omega \varepsilon \Omega$, $T_\omega(E) \subset T_\omega(F)$;

(c) If $E \in \mathcal{L}^1$ and $\omega_n \to \omega_0$ (in Ω), then

$$\lim_{n\to\infty} |T_{\omega n}(E)| = |T_{\omega_0}(E)| = |E|.$$

Example 1. Set Ω equal to the real line R^1 equipped with the Euclidean metric. If $E \in \mathcal{L}^1$ let $T_\omega(E) = E + \omega$ (i.e. the set of all real numbers of the form $x + \omega$, $x \in E$). Taking 0 as ω_0 it is easy to see that properties (a), (b), and (a) are satisfied.

Example 2. Set Ω equal to the interval $(0, 1\rangle$ equipped with the Euclidean metric. If $E \in \mathcal{L}$, then for $\omega \in (0, 1\rangle$, let $T_\omega(E) = \omega E$ (i.e. the set of all real numbers of the form ωx, $x \in E$). If we put $\omega_0 = 1$ then properties (a), (b), and (c) are satisfied.

M. Pal [3] considered an extension of the families of transformations of Neubrunn and Šalát, namely for each ω belonging to a metric space Ω he associated a Transformation T_ω, mapping $\mathcal{L}^n$ (the collection of measurable subsets of R^n (n-dimensional Euclidean space)) into $\mathcal{L}^n$ in such a way that the family of transformations $\{T_\omega\}_{\omega \varepsilon \Omega}$ satisfies the following three conditions.

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(I) There exists $\omega_0 \in \Omega$ such that for every closed sphere $K = S[a, r] \subset R^n$ and every sequence $\{\omega_n\}$ $(\omega_n \in \Omega)$ converging to ω_0,

$$\lim_{n\to\infty} [\sup \{ | a - T_{\omega_n}(K) | \}] = r \text{ holds.}$$

(II) If $E, F \in \mathcal{L}^n$ and $F \subset E$, then for every $\omega \in \Omega$, $T_\omega(F) \subset T_\omega(E)$.

(III) If $E \in \mathcal{L}^n$ and $\omega_n \to \omega_0$, then

$$\lim_{n\to\infty} | T_{\omega_n}(E) | = | T_{\omega_0}(E) | = | E |.$$

Saha and Ray [4] on page 238 of their work consider a family of transformations $\{T_\omega\}_{\omega\in\Omega}$ of $\mathcal{L}^n$ int $\mathcal{L}^n$ which they require to satisfy the following three conditions.

(i) There exists $\omega_0 \in \Omega$ such that for any two spheres $K_1 = S[a, r_1]$ and $K_2 = S[b, r_2]$ in R^n and every sequence $\omega_n \in \Omega$ converging to ω_0.

$$\lim_{n\to\infty} [\sup \{ | a - T_{\omega_n}(K_2) | \}] = \min(r_1, r_2) \text{ if } r_1 \neq r_2$$

$$= r \text{ if } r_1 = r_2 = r.$$

(ii) If E and F are two measurable sets in R^n such that $F \subset E$, then for every $\omega \in \Omega$, $T_\omega(F) \subset T_\omega(E)$.

(iii) If E is a measurable set in R^n and $\omega_n \to \omega_0$ (in Ω),

$$\lim_{n\to\infty} | T_{\omega_n}(E) = | T_{\omega_0}(E) | = | E |.$$

Here $|a - B|$ denotes the set $\{ |a - b|;\ b \in B\}$ where $|a - b|$ is the ordinary Euclidean distance between a and b.

Clearly, no family of transformations $\{T_\omega\}_{\omega\in\Omega}$ can satisfy condition (i) as it stands for several reasons, the first and most obvious reason being that the expression $\lim_{n\to\infty} [\sup \{ | a - T_{\omega_n}(K_2) | \}]$ is dependent on a and b (as well as r_2), while the expression $\min(r_1, r_2)$ is independent of a and b. Secondly, $\lim_{n\to\infty} [\sup \{ | a - T_{\omega_n}(K_2) | \}]$ depends on a, b, and r_2 but not on r_1, so that it being equal to the $\min(r_1, r_2)$ is absurd, for if $r_1 < r_2 < r_1'$ and we take $K_1 = S[a, r_1]$ and $K_2 = S[b, r_2]$ then we get by (i) that the $\lim [\sup | a - T_{\omega_n}(K_2) | \}] = r_1$, while if we take $K_1 = S[a, r_1']$ and $K_2 = S[b, r_2]$, then we get by (i) that the $\lim_{n\to\infty} [\sup \{ | a - T_{\omega_n}(K_2) | \}] = r_2$.

Because of these difficulties (i) should be ammended to read as follows.

(i)′ There exists a papir of poits (in R^n) a and b such that

$$\lim_{n\to\infty} [\sup \{ | a - T_{\omega_n}(K) | \}] = r \text{ for every sphere } K = S[b, r],\ r > 0.$$

Theorem 1 of Saha and Ray should be stated as follows.

Theorem 1′ *Supose A and B are two sets of positive measure in R^n and a is a point of density one in A, b is a point of density one in B and ω_0 is a point in Ω. Supose $\{T_\omega\}_{\omega\varepsilon\Omega}$ is a family of transformations of $\mathcal{L}^n$ into $\mathcal{L}^n$ satisfying the properties* (i)′, (ii) *and* (iii) *with respect to the points a, b, and ω_0 mentioned above, then there exists a natural number N_0, such that for $n \geqslant N_0$ the set $A \cap T_{\omega_n}(B)$ has positive Lebesgue measure.*

The proof of Theorem 1 of Saha and Ray is a proof of Theorem 1′.

More work is required on Theorem 2, for here a mere restatement of the theorem will not su.fice, a new proof is also needed. Theorem 2 should reed as follows.

Theorem 2′. *Suppose A and B are two sets of positive measure in R^n and a is a point of density one in A, b is a point of density one in B and ω_0 is a point of Ω. Suppose $\{T_\omega\}_{\omega\varepsilon\Omega}$ is a family of transformbtions of $\mathcal{L}^n$ into $\mathcal{L}^n$ satisfying the properties* (i)′, (ii) *and* (iii) *with respect to the points a, b, and ω_0 mentioned above, then if $\{\omega_n\}_{n=1}^{\infty}$ is a sequence in Ω converging to ω_0 and p is a positive integer, then there exists p strictly increasing integers $n_1, n_2, \ldots, n_p$ such that*

$$A \cap T_{\omega_{n_1}}(B) \cap T_{\omega_{n_2}}(B) \cap \cdots \cap T_{\omega_{n_p}}(B)$$

is a set of positive measure.

The proof of saha and Ray (page 240) breaks down, because while the set $C_1 = A \cap T_{\omega_{n_1}}(B)$ has positive measure it may turn out that a is not a point of densty one of C_1 and hence Theorem 1′ can not be applied to the pair of sets C_1 and B.

We offer the following proof of Theorem 2′.

Proof of Theorem 2′. Let $0<\varepsilon<1$, then since a and b are density points there exists $r_\varepsilon>0$ such that

$$(1) \quad \begin{aligned} &|S[a, r_\varepsilon]| - |A \cap S[a, r_\varepsilon]| < \varepsilon |S[a, r_\varepsilon]| \\ &|S[b, r_\varepsilon]| - |B \cap S|[b, r_\varepsilon]| < \varepsilon |S[b, r_\varepsilon]| \end{aligned}$$

By (ii) and (iii) there exists N_ε such that $n \geqslant N_\varepsilon$ implies

$$(2) \quad |T_{\omega_n}(S[b,r_\varepsilon]) \setminus T_{\omega_n}(S[b, r_\varepsilon]) \cap B)| \leqslant |S[b, r_\varepsilon]| - |S[b, r_\varepsilon] \cap B| + \\ + \varepsilon |S[b, r_\varepsilon]| < 2\varepsilon |S[b, r_\varepsilon]| \quad \text{(by (1))}.$$

By (i)′ and (iii) there exists $N_\varepsilon' > N_\varepsilon$ such that

$$(3) \quad |T_{\omega_n}(S[b, r_\varepsilon]) \cap S[a, r_\varepsilon]| > (1-\varepsilon) \cdot |S[a, r_\varepsilon]| \text{ for every } n \geqslant N_\varepsilon'.$$

By (2) and (3) we get

$$4) \quad |T_{\omega_n}(S[b, r_\varepsilon] \cap B) \cap S[a, r_\varepsilon]| > (1-3\varepsilon) \cdot |S[a, r_\varepsilon]| \text{ for every } n \geqslant N_\varepsilon'.$$

For each $i=1, 2, \ldots, p$, set $\varepsilon_i' = 1/6 \cdot 2^i)$.

Then if $0<\varepsilon<\varepsilon_i'$ we have

$$(5) \quad |T_{\omega_n}(S[b, r_\varepsilon] \cap B) \cap S[a, r_\varepsilon]| > (1 - 1/(2 \cdot 2^i)) \cdot |S[a, r_\varepsilon]| \text{ if } n \geqslant N_\varepsilon' \text{ and}$$

$$(6) \quad |S[a, r_\varepsilon]| - |A \cap S[a, r_\varepsilon]| < (1/(2 \cdot 2^i) |S[a, r_\varepsilon]| \text{ if } 0<\varepsilon<\varepsilon_i'.$$

From (5) and (6) it follows that

(7) $$|T_{\omega_n}(S[b, r_\varepsilon]\cap B)\cap(A\cap S[a, r_\varepsilon])|>(1-1/2^i)\cdot|S[a, r_\varepsilon]|$$

if $0<\varepsilon<\varepsilon_i'$ and $n\geqslant N_\varepsilon'$.

Let ε be a fixed real number, $0<\varepsilon<\min(\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_p')$ and let $n_1, n_2, \ldots, n_p$ be p distinct integers each larger then N_ε', then it follows that

(8) $$|T_{\omega_{n_i}}(S[b, r_\varepsilon]\cap B)\cap(A\cap S[a, r_\varepsilon])|>(1-1/2^i)\cdot|S[a, r_\varepsilon]|$$

for each $i=1, 2, \ldots, p$.

From this it follows that the set

(9) $$\left\{\bigcap_{i=1}^{p} T_{\omega_{n_i}}(S[b, r_\varepsilon]\cap B)\right\}\cap(A\cap S[a, r])$$

has positive measure, completing the proof.

Theorem 3 of Saha and Ray should be stated as follows.

Theorem 3′. *Suppose $A, B_1, B_2, \ldots, B_m$ are sets of positive measure in R^n and a is a point of density one in A, b_i is a point of density one in B_i for each $i=1, 2, \ldots, m$ and ω_0^i is a point of Ω for each $i=1, 2, \ldots, m$. Suppose $\{T_\omega\}_{\omega\in\Omega}$ is a family of transformations on $\mathcal{L}^n$ into $\mathcal{L}^n$ satisfying the properties* (i)′, (ii) *and* (iii) *with respect to the triple (a, b_i, ω_0^i) for each $i=1, 2, \ldots, m$. If the sequence $\{\omega_n^i\}_{n=1}^\infty$ converges to ω_0^i for each $i=1, 2, \ldots, m$, then there exists a positive integer N such that for $n\geqslant N$,*

$$A\subset T_{\omega_n}1(B_1)\cap T_{\omega_n}2(B_2)\cap\cdots\cap T_{\omega_n}m(B)$$

is a set of posttive measure.

The proof of Saha and Ray of this result (page 241) breaks down, again because Theorem 1′ is not applicable to the pair of sets C_1 and B_2. However Theorem 3′ ist true and its proof is similar to the proof of Theorem 2′ given above and will therefore be omitted.

The proof of Theorem 4 of Saha and Ray also breaks down. This theorem and several other results about transformations of sets in R^n are the subject of a recently written paper [1] of the current author.

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