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ON SYMMETRIC WORDS IN NILPOTENT GROUPS

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Symmetric words (operations) in various groups were investigated by E. Płonka ([2], [3], [4]). Most of the notions and notation we shall use are from these Płonka's articles.

Let r be a positive integer and S_r the permutation group of the set $\{1, \ldots, r\}$. A group word $w = w(x_1, \ldots, x_r)$ is called *symmetric* in the group G if

$$w(a_1, \ldots, a_r) = w(a_{\sigma 1}, \ldots, a_{\sigma r})$$

for every $a_1, \ldots, a_r \in G$ and every $\sigma \in S_r$.

Let $F_G(x_1, \ldots, x_r)$ be the group freely generated by $x_1, \ldots, x_r$ in the smallest variety of groups containing G. Let A be the group of automorphisms of $F_G(x_1, \ldots x_r)$ induced by the mappings

$$x_i \rightarrow x_{\sigma i}, \quad 1 \leqslant i \leqslant r,$$

where $\sigma \in S_r$. The set

$$S^{(r)}(G) = \{w \in F_G(x_1, \ldots, x_r) \mid w = \alpha w \text{ for every } \alpha \in A\}$$

is just the set of all symmetric words in G of r variables $x_1, \ldots, x_r$. $S^{(r)}(G)$ is a group. In [3] and [4] it is completely described in the case of a free nilpotent G or any G of nilpotency class $\leqslant 3$.

Clearly, the mapping $\partial^r_{r-1} : S^{(r)}(G) \rightarrow S^{(r-1)}(G)$ defined by

$$\partial^r_{r-1}(w(x_1, \ldots, x_r)) = w(x_1, \ldots, x_{r-1}, 1)$$

is a homomorphism. Furthermore, Płonka has proved that ∂^r_{r-1} is in fact an isomorphism when G is a free nilpotent group of class n and $r > n$ ([3]), and also when G is any nilpotent group of class $n \leqslant 3$ and $r > n$ ([4]). These results suggest naturaly a more general problem. formulated in [4]: is the mapping ∂^r_{r-1} an isomorphism for any nilpotent group G and any r greater than the nilpotency class of G? In this paper we give a positive solution to this problem.

Let us assume that G is a nilpotent group of class n and that $r>n$. Let N be the free nilpotent group of class n with generators $x_1, \dots, x_r$ and let N' be the subgroup of N generated by $x_1, \dots, x_{r-1}$. N' is also a free nilpotent group of class n and $x_1, \dots, x_{r-1}$ are its free generators. There exists a fully invariant subgroup U of N such that $F_G(x_1, \dots, x_r)$ is isomorphic to N/U. Let us denote by φ_i, $1 \leqslant i \leqslant r$, the endomorphism of N induced by the mapping

$$x_j \to \begin{cases} x_i, & \text{for } j=i \\ 1, & \text{for } j \neq i \end{cases}$$

and let

$$w_{ij\dots k} = \varphi_i \varphi_j \dots \varphi_k (w),$$

for any $w \in N$ and $1 \leqslant i, j, \dots, k \leqslant r$. Clearly

$$F_G(x_1, \dots, x_{r-1}) \cong N'/U',$$

where $U' = \{w_r \mid w \in U\}$.

Finally, for any $i, j, \dots k \in \{1, \dots, r\}$, let us define a transformation $F_{ij\dots k}$ of the group N as follows:

$$F_i w = w \cdot w_r^{-1},$$

$$F_{ij\dots k} w = F_i (F_{j\dots k} w).$$

In view of this definition we obtain that the equality

$$(1) \qquad w = F_{12\dots r} w \cdot F_{2\dots r} w_1 \cdot F_{3\dots r} w_2 \cdots F_r w_{r-1}$$

holds for every $w \in N$.

Lemma 1. Let w be an element of N; then

(a) $F_{12\dots r} w = 1$,

(b) if $w_i \in U$ for every $i \in \{1, \dots, r\}$, then $w \in U$.

Proof. (a) The identity $F_{12\dots r} w = 1$ follows immediately from the statements 33.38. and 33.42. of [1]. We only note that without the assumption $r>n$ this identity may not be true.

(b) Using (a) of this Lemma and (1) we obtain

$$(2) \qquad w = F_{2\dots r} w_1 \cdot F_{3\dots r} w_2 \cdots F_r w_{r-1} \cdot w_r$$

for every $w \in N$.

From the definition of $F_{ij\dots k}$ and the fact that U is fully invariant it easily follows that if $u \in U$, then $F_{ij\dots k} u \in U$. So, on the right hand side of the equality (2) we have a product of the elements of U and hence $w \in U$.

Lemma 2. Let u^i, $1 \leqslant i \leqslant r$, be elements of N which satisfy the conditions

$$\varphi_i(u^j) \equiv \varphi_j(u^i) \pmod U$$

for every $i, j \in 1, \dots, r\}$. Then there exists a $w \in N$ such that $\varphi_i(w) \equiv u^i \pmod U$ holds for every $i \in \{1, \dots, r\}$.

Proof. Let w be an element of N defined by

$$w = F_{2\dots r}u^1 \cdot F_{3\dots r}u^2 \cdots F_r u^{r-1} \cdot u^r.$$

For every $i \in \{1, \dots, r\}$ we hawe

$$\varphi_i(w) = F_{2\dots r}u_i^1 \cdot F_{3\dots r}u_i^2 \cdots F_r u_i^{r-1} \cdot u_i^r. \tag{3}$$

From the definition of $F_{ij\dots k}$ it follows that if $u \equiv v \pmod U$, then $F_{ij\dots k}u \equiv \equiv F_{ij\dots k}v \pmod U$. So, from the assumptions $u_q^p \equiv u_p^q$, we obtain

$$F_{ij\dots k}u_q^p \equiv F_{ij\dots k}u_p^q \pmod U$$

for every $i, j, \dots, k, p, q \in \{1, \dots, r\}$. Applying this to (3) gives

$$\varphi_i(w) \equiv F_{2\dots r}u_1^i \cdot F_{1\dots r}u_2^i \cdot \dots \cdot F_r u_{r-1}^i \cdot u_r^i \pmod U.$$

Now (2) implies required congruence

$$\varphi_i(w) \equiv u^i \pmod U.$$

Theorem. *Let G be a nilpotent group of class n. For every $r > n$ the mapping ∂_{r-1}^r is an isomorphism.*

Remark. Examples from [4] demonstrate indispensability of the assumtion $r > n$.

Proof. We may regard ∂_{r-1}^r as the restriction of the mapping $wU \to w_r U'$ (from N/U into N'/U') on the set $S^{(r)}(G) \subseteq N/U$. Since ∂_{r-1}^r is a homomorphism, it remains to show that it is "1 – 1" and "onto".

1° ∂_{r-1}^r *is "1 – 1"*. Let $u, v \in N$ and $uU, vU \in S^{(r)}(G)$. Let also $\partial_{r-1}^r(uU) = \partial_{r-1}^r(vU)$. The latter condition is equivalent to

$$u_r \equiv v_r \pmod{U'}$$

and also to

$$u_r \equiv v_r \pmod U.$$

Hence, using the fact that U is fully invariant and that $uU, vU \in S^{(r)}(G)$ we can deduce

$$u_i \equiv v_i \pmod U$$

for every $i \in \{1, \dots, r\}$. Thus $(uv^{-1})_i = u_i v_i^{-1} \in U$ for every i. By the Lemma 1 (b) we conclude

$$uU = vU.$$

2° ∂_{r-1}^r *is "onto"*. Let $uU' \in S^{(r-1)}(G)$, where $u = u(x_1, \dots, x_{r-1}) \in N' \subset N$. Let us define the elements $u^i \in N$ by

$$u_i = u(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r).$$

It is not difficult to see that from $uU' \in S^{(r-1)}(G)$ it follows that

$$\varphi_i(u^j) \equiv \varphi_j(u^i) \pmod U$$

holds for every $i, j \in \{1, \ldots, r\}$. Now we can apply Lemma 2 which assures the existence of a $w \in N$ such that

$$\varphi_i(w) \equiv u^i \pmod{U}$$

It remains to prove that $wU \in S^{(r)}(G)$.

Let α be an automorphism from A; then $\alpha w = w(x_{\sigma 1}, \ldots, x_{\sigma r})$ for a certain permutation $\sigma \in S_r$. Hence,

$$(\alpha w)_i = w(x_{\sigma 1}, \ldots, x_{\sigma(j-1)}, 1, x_{\sigma(j+1)}, \ldots, x_{\sigma r}),$$

where $\sigma j = i$; and further

$$\begin{aligned}(\alpha w)_i &\equiv u^j(x_{\sigma 1}, \ldots, x_{\sigma(j-1)}, x_{\sigma(j+1)}, \ldots, x_{\sigma r}) \\ &= u(x_{\sigma 1}, \ldots, x_{\sigma(j-1)}, x_{\sigma(j+1)}, \ldots, x_{\sigma r}) \\ &\equiv u(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r) \\ &= u^i \pmod{U}\end{aligned}$$

Applying this congruence we obtain

$$\varphi_i(w(\alpha w)^{-1}) = w_i((\alpha w)_i)^{-1} \equiv u^i(u^i)^{-1} = 1 \pmod{U}$$

for every $i \in \{1, \ldots, r\}$. Now from Lemma 1(b) we deduce

$$w \equiv \alpha w \pmod{U}$$

Since this is true for every $\alpha \in A$, we conclude that wU is an element of $S^{(r)}(G)$.

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