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ON 0-MINIMAL BI-IDEALS OF SEMIGROUPS

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In this paper the concept of a degenerate bi-ideal of a semigroup S $(S=S^0)$ has been introduced. If B is a 0-minimal bi-ideal of S, then either B is a degenerate bi-ideal or $bSb=B$ for every b in $B\setminus\{0\}$. As a corollary, we have a result of K. Kapp (Theorem 1). Also, as a consequence of some results for 0-minimal bi-ideals of S, we have: if $S\neq S^0$, then a non-empty subset B of S is a minimal bi-ideal of S if and only if B is a minimal right ideal of some minimal left ideal of S. Corollary 3 gives a characterization of a completely 0-simple semigroups by a non-degenerate 0-minimal bi-ideal.

The reader is refered to [1] and [2] for all concepts not defined in the paper.

A bi-ideal B of a semigroup $S\,(S=S^0)$ is called *degenerate* if $B=\{0, b\}$ with $bS^1b=\{0\}$.

Lemma 1. Let $S=S^0$ and let B be a 0-minimal bi-ideal of S. Then either $bSb=B$ for every b in $B\setminus\{0\}$ or B is degenerate.

Proof. Let $b\in B\setminus\{0\}$. Then $bSb\subset B$ and bSb is a bi-ideal of S. Therefore, either $bSb=\{0\}$ or $bSb=B$. Let $bSb=\{0\}$. If $b^2=b$ then $b^3=b$. This is impossible, since $b^3\in bSb$. Let $b^2\in B\setminus\{0, b\}$. Thus $\{0, b, b^2\}\subset B, \{0, b^2\}S^1\{0, b^2\}=$ $=\{0\}$. Therefore, $\{0, b^2\}$ is a bi-ideal of S properly contained in B. This is impossible because B is a 0-minimal bi-ideal of S. We have $b^2=0$ and $B=\{0, b\}$ with $bS^1b=\{0\}$ so that B is a degenerate bi-ideal.

It is easy to see that the following s atements holds:

Lemma 2. A subset B of a semigroup $S=S^0$ is a non-degenerate 0-minimal bi-ideal of S if and only if $B=bSb$ for every b in $B\setminus\{0\}$.

Lemma 3. The following conditions on a semigroup S with zero 0 are equivalent:

i) S is a group with zero 0.

ii) $aSa=S$ for every $a\in S\setminus\{0\}$.

iii) $a_1Sa_2=S$ for every $a_1, a_2\in S\setminus\{0\}$.

As a consequence we have a result of K. Kapp [3].

Theorem 1. *Let $S=S^0$ and let B be a 0-minimal bi-ideal of S. Then either $B^2=\{0\}$ or B is a group with zero* 0.

Proof. Assume $B^2\neq\{0\}$. Let $a\in B\setminus\{0\}$. By Lemma 1, $aSa=B$. Since aBa is a bi-ideal of S contained in B, it follows that either $aBa=\{0\}$ or $aBa=B$. Assume that $aBa=\{0\}$. Thus $\{0\}=aBa=a(aSa)a=a^2Sa^2$. According to Lemma 1, we have $a^2=0$. Then $B^2=(aSa)(aSa)=aSa^2Sa=\{0\}$. This is impossible. Therefore $aBa=B$. By Lemma 3, B is a group with zero.

S. Lajos [4] showed that a subset B of a semigroup S without zero is a bi-ideal of S if and only if B is a right ideal of some left ideal of S. Now we give an analogous statement for minimal bi-ideals of a semigroup S, as a consequence of some results for 0-minimal bi-ideals of a semigroup with zero 0.

Lemma 4. Let $S=S^0$ and let B be a non-degenerate (degenerate) 0-minimal right ideal of some 0-minimal left ideal L of S. Then B is a non-degenerate (degenerate) 0-minimal bi-ideal of S.

Proof. If L is a degenerate 0-minimal left ideal of S then $B=L$ and B is a degenerate 0-minimal bi-ideal of S. Let L be a non-degenerate 0-minimal left ideal of S. Let B be a degenerate 0-minimal right ideal of L. Then $B=\{0, b\}$ with $bL=\{0\}$ and $Sb=L$. Therefore $bS^1b=\{0\}$ i.e. B is a degenerate 0-minimal bi-ideal of S. Let B be a non-degenerate 0-minimal right ideal of L. Then $Sb=L$ and $bL=B$ for every $b\in B\setminus\{0\}$. Thus $B=bSb$ for every b in $B\setminus\{0\}$. By Lemma 2, B is a non-degenerate 0-minimal bi-ideal of S.

Lemma 5. Let $S=S^0$. Let B be a non-degenerate 0-minimal bi-ideal of S and $b\in B\setminus\{0\}$. Then

i) $B=bSb$.

ii) If L is a left ideal of S contained in Sb, then $L^2=\{0\}$ or $L=Sb$.

iii) If R is a right ideal of Sb contained in B, then $R^2=\{0\}$ or $R=B$.

Proof. i). By Lemma 1, $B=bSb$.

ii) Let $L\subset Sb$ and $SL\subset L$. Thus $bL\subset bSb=B$ and bL is a bi-ideal of S contained in B. It follows that $bL=\{0\}$ or $bL=B$. If $bL=\{0\}$, then $SbL=\{0\}$. Since $L\subset Sb$ we have $L^2=\{0\}$. If $bL=B$, then $bL\subset L$ implies $B\subset L$. Thus $Sb\subset SL\subset L$ i.e. $L=Sb$.

iii) Let $R\subset B$ and $RSb\subset R$. Since RSb is a bi-ideal of S, contained in B, it follows that $RSb=\{0\}$ or $RSb=B$. If $RSb=\{0\}$ then $RB=\{0\}$. Since $R\subset B$ we have $R^2=\{0\}$. If $RSb=B$ then $RSb\subset R$ implies $B\subset R$ i.e. $B=R$.

Corollary 1. Let $S=S^0$ and let B be a 0-minimal bi-ideal of S such that B is a group with zero 0. Let $b\in B\setminus\{0\}$. Then

i) $B=bSb$.

ii) If L is a left ideal of S contained in Sb, then either $L^2=\{0\}$ or $L=Sb$.

iii) B is a 0-minimal right ideal of Sb.

According to Lemma 2.34 [1] and Lemma 5 we have

Corolary 2. Let $S=S^0$ an let B be a non-degenerate 0-minimal bi-ideal of S. Let $b\in B\setminus\{0\}$. Then

i) $B=bSb$.

ii) If P is a 0-simple subsemigroup of S such that $Sb\subset P$ then Sb is a 0-minimal left ideal of S.

iii) If T is a 0-simple subsemigroup of S such that $B\subset T\subset Sb$ then B is a 0-minimal right ideal of Sb.

By Lemma 4 and Corollary 1 we immediately obtain the following theorem.

Theorem 2. *Let $S\neq S^0$ and $\varnothing\neq B\subseteq S$. Then B is a minimal bi-ideal of S if and only if B is a minimal right ideal of some minimal left ideal of S.*

Nov we describe a completely 0-simple ideal of a semigroup $S=S^0$, by 0-minimal bi-ideals.

Theorem 3. *Let M be an ideal of $S=S^0$. Then M is completely 0-simple if and only if $M^2\neq\{0\}$ and M is 0-minimal ideal of S which contains at least one non-degenerate 0-minimal bi-ideal of S.*

Proof. $\Leftarrow$ Let M be a 0-minimal ideal of S such that $M^2\neq\{0\}$. Assume that M contains a non-degenerate 0-minimal bi-ideal B of S. Let $b\in B\setminus\{0\}$. Then M is 0-simple (Th. 2.29 [1]) and $Sb\subset M$, $bS\subset M$. According to Corollary 2, Sb and bS are 0-minimal left and 0-minimal right ideals of S, respectively. Therefore, M is completely 0-simple (Cor. 2.50 [1]). $\Rightarrow$ Let M be a completely 0-simple ideal of S. Then $M^2\neq\{0\}$, M is a 0-minimal ideal of S and M contains at least one non-degenerate 0-minimal bi-ideal B of M. Let $b\in B\setminus\{0\}$. According to Corollary 2, bM and Mb are 0-minimal right and 0-minimal left ideal of S, respectively, and $bMMb=bMb=B$. Therefore B is a 0-minimal bi-ideal of S [2].

Corollary 3. Let S be a semigroup with zero 0. Then S is completely 0-simple if and only if S is 0-simple and S contains at least one non-degenerate 0-minimal bi-ideal of S.

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