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ON NEARLY STRONGLY PARACOMPACT SPACES*

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In a paper [1] the author has introduced and studied the class of nearly strongly paracompact spaces.. A space X is said to be *nearly strongly paracompact* (NSPC) iff every regular open cover of X admits a star finite, open refinement. The class of nearly strongly paracompact spaces contains the class of strongly paracompact spaces. In the present paper, it is proposed some further results on nearly strongly paracompact spaces.

Notation is standard except that $\alpha(A)$ will be used to denote the interior of the closure of A. The topology τ^* is the semi regularization of τ ([2]) and has as base the regularly open sets from τ. A closed set F of X is said to be star-closed if F is closed in (X, τ^*).

1. Characterizations

Lemma 1.1. *Every regular open cover* $\mathcal{U}=\{U_\alpha : \alpha\in I\}$ *of a space* X *has an open star finite refinement if and only if* $\mathcal{U}$ *has a regular open star finite refinement.*

Proof. Let $\mathcal{U}$ be any regular open cover of X. Then, there exists a star finite open refinement $\mathcal{V}$ of $\mathcal{U}$. Consider the family

$$\mathcal{W}=\{\alpha(V) : V\in\mathcal{V}\}.$$

Then, $\mathcal{W}$ is a star finite regular open refinement of $\mathcal{U}$.

Theorem 1.1. *A space* (X, τ) *is nearly strongly paracompact iff* (X, τ^*) *is strongly paracompact.*

Proof. Let (X, τ) be a nearly strongly paracompact space. Let $\mathcal{C}$ be any basic τ^*-open covering of X. Then, $\mathcal{C}$ is also a τ-regular open covering and hence there exists a τ-regular open star finite refinement $\mathcal{V}=\{V_\beta : \beta\in J\}$ of $\mathcal{C}$. Hence (X, τ^*) is srongly paracompact.

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Conversely, let $\mathscr{C}$ be a τ-regular open covering of X. Then $C=\alpha(C)$ for all $C\in\mathscr{C}$ and therefore $\{\alpha(C):C\in\mathscr{C}\}$ is a regular open covering, hence there exists a star finite τ^*-open refinement $\mathscr{V}=\{V_\beta:\beta\in J\}$.

Then, $\mathscr{V}$ is a star finite τ-open refinement of a family $\mathscr{C}$ and hence (X, τ) is nearly strongly paracompact.

Definition 1.1. A space X is said to be *almost regular* iff for any regularly closed set F and any point $x\notin F$, there exist disjoint open sets containing F and x respectively, [3].

Theorem 1.2. *For an almost regular space* (X, τ), *the following are equivalent:*

(a) (X, τ) *is nearly strongly paracompact,*

(b) *Every regular open cover of* (X, τ) *has an open star countable refinement,*

(c) *Every regular open cover of* (X, τ) *has a regular open star countable refinement,*

(d) *Every regular open cover of* (X, τ) *has a regular open star finite refinement,*

(e) *Every regular open cover of* (X, τ) *has a regular closed refinement which is both locally finite and star finite,*

(f) *Every regular open cover of* (X, τ) *has a regular closed refinement which is both locally finite and star countable.*

Proof. $(a)\to(b)$ Obvious.

$(b)\to(c)$ It is similar to the proof of Lemma 1.1.

$(c)\to(d)$ Let (X, τ) be an almost regular space. Let $\mathscr{U}$ be any regular open cover of X. Then, (X, τ^*) is a regular space and the τ^*-open cover $\mathscr{U}$ has a τ^*-open, star countable refinement $\mathscr{V}$. Therefore, by Theorem 1 in [4], there exists a τ^*-open star finite refinement $\mathscr{V}'$ of $\mathscr{U}$. Consider the family

$$\mathscr{V}^*=\{\alpha(V):V\in\mathscr{V}'\}.$$

Then, this is a star finite, regularly open refinement of $\mathscr{U}$, and hence the result.

$(d)\to$(e) Let $\mathscr{U}$ be any regular open cover of (X, τ). By hypothesis there exists a star finite, regular open refinement $\mathscr{V}$ of $\mathscr{U}$. Since (X, τ^*) is regular, there exists a star finite τ^*-closed refinement $\mathscr{F}$ of $\mathscr{U}$ (Theorem 1, [4]). Consider the family

$$\mathscr{F}^*=\{\overline{F^0}:F\in\mathscr{F}\}.$$

Clearly, $\mathscr{F}^*$ is a family of regulary closed sets, which refines $\mathscr{U}$ and it is both locally finite and star finite.

By lemma 1.1 in [3], $\mathscr{F}^*$ is a cover of the space X.

$(e)\to(f)$ Obvious.

$(f)\to(b)$ Let $\mathscr{U}=\{U_\lambda:\lambda\in\Lambda\}$ be any regular open covering of (X, τ). Then, there exists a regular closed refinement $\mathscr{F}=\{F_\lambda:\lambda\in\Lambda\}$ which is both locally finite and star countable. Let $\mathscr{F}=\bigcup\{\mathscr{F}_\beta:\beta\in J\}$, where the $\mathscr{F}_\beta$'s are the components of $\mathscr{F}$. It follows from Lemma 2 in [4], that all families $\mathscr{F}_\beta$ are countable; let $\mathscr{F}_\beta=\bigcup\{F_{\beta,i}:i=1, 2, \dots, n, \dots\}$. The sets $C_\beta=\bigcup\mathscr{F}_\beta$ are pairwise disjoint (Lemma 1, [4]) and — by the local finitness of $\mathscr{F}$ — are cloced-and-open. For every $\beta\in J$ and any natural number i take $\lambda(\beta, i)\in\Lambda$ such that $F_{\beta,i}\subset U_{\lambda(\beta, i)}$.

The family

$$\{C_\beta \cap U_{\lambda(\beta,i)} : i=1, 2, \dots n, \dots, \beta \in J\}$$

is a star countable open refinement of $\mathcal{U}$.

$(d) \to (a)$ Obvious.

Corollary 1.1. *Let (X, τ) be an almost regular nearly strongly paracompact space. Then, every regular open cover $\mathcal{U}$ of (X, τ) admits a τ^*-open refinement $\mathcal{V}$, whose members are F_σ' s in τ^*.*

Proof. It follows easily from Theorem 1.2. and Theorem 1 in [4].

Lemma 1.2. *Let X be a topological space and let $\mathcal{A}$ be any open locally finite covering of X, the closures of whose members are α-nearly compact subsets (a subset A is α-nearly compact if every regular open (in X) cover of A has a finite subcovering). Then $\mathcal{A}$ is a star finite open covering of X.*

Proof. Let A be any member of the family $\mathcal{A}$. Since $\mathcal{A}$ is a locally finite family, for each point $x \in \bar{A}$, there exists an open set O_x such that $x \in O_x$ and O_x intersects finitely many members of $\mathcal{A}$. Then, $\alpha(O_x)$ is a regular open set containing x which intersects finitely many members of $\mathcal{A}$, because for any $A \in \mathcal{A}$, $A \cap O_x = \varnothing \Rightarrow A \cap \alpha(O_x) = \varnothing$. Consider the regular open covering $\{\alpha(O_x) : x \in \bar{A}\}$ of $\bar{A}$. Since the set $\bar{A}$ is α-nearly compact, there exists a finite subfamily $\{\alpha(O_{x(i)}) : i \in 1, 2, \dots, s\}$ such that $\bar{A} \subset \cup\{\alpha(O_{x(i)}) : i=1, 2, \dots, s\}$. Since every set $\alpha(O_{x(i)})$ intersects finitely many members of $\mathcal{A}$, than the set $O = \bigcup_{i=1}^{s} \alpha(O_{x(i)})$ also intersects finitely many members of $\mathcal{A}$. Since $A \subset O$, the set A intersects finitely many members of $\mathcal{A}$, hence $\mathcal{A}$ is a star finite family.

Theorem 1.3. *Let X be any locally nearly compact and nearly paracompact space. Then X is nearly strongly paracompact.*

Proof. Let $\mathcal{A}$ ba any regular open covering of X. Since X is locally nearly compact, for each point $x \in X$, there exists a regular open set O_x containing x, such that $\bar{O}_x$ is an α-nearly compact subset of X. Let $\mathcal{A}^* = \{O_x : x \in X\}$. Consider the regular open covering $\mathcal{A}' = \{A \cap A^* : A \in \mathcal{A}$ and $A^* \in \mathcal{A}^*\}$. Since X is nearly paracompact, there exists a locally finite regular open refinement $\mathcal{A}''$ of $\mathcal{A}'$. For any $A'' \in \mathcal{A}''$, there exists $A^* \in \mathcal{A}^*$, such that $A'' \subset A^*$, i. e. $\bar{A}'' \subset \bar{A}^*$. Since $\bar{A}^*$ is α-nearly compact subset of X, then $\bar{A}$ is also α-nearly compact subset of X. Then by preceeding lemma, $\mathcal{A}''$ is a star finite family. Hence X is nearly strongly paracompact.

A mapping $f : X \to Y$ is said to be *almost continuous* iff the inverse image of every regulary open subset of Y is an open subset of X. f is called *almost open (almost closed, star closed)* iff the image of every regularly open (regularly closed, star closed) subset of X is an open (a closed) subset of Y [3].

Theorem 1.4. *If f is star closed, almost continuous and almost open mapping of a nearly strongly paracompact space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, then Y is nearly strongly paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any regular open covering of Y. Since f is almost continuous and almost open mapping, then $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a regular open covering of X. Since X is nearly strongly paracompact, there exists a star finite, regular open refinement $\mathcal{V}=\{V_\beta:\beta\in J\}$ of $\{f^{-1}(U_\beta):\alpha\in I\}$. Since f is almost open $\{f(V_\beta):\beta\in J\}$ is an open covering of Y. Let

$$V^*(\beta_1, \beta_2, \ldots, \beta_k)=\bigcap_{i=1}^{k} f(V_{\beta i})\cap(Y\setminus f(X\setminus\bigcup_{i=1}^{k} V_{\beta i})),\ \beta_1, \beta_2, \ldots \beta_k\in J.$$

Since f is star closed and $f^{-1}(y)$ is α-nearly compact for each $y\in Y$, then the family

$$\mathcal{V}^*=\{V^*(\beta_1, \beta_2, \ldots \beta_k):\beta_1, \beta_2, \ldots \beta_k\in J\}$$

is an open star finite covering of Y.

The family $\mathcal{V}^*$ is open star finite refinement of $\mathcal{U}$, hence Y is nearly strongly paracompact.

Lemma 1.3. *Let f be any almost open and almost closed mapping of a space X onto a space Y such that $f^{-1}(G)$ is α-nearly compact for every proper open subset $G\subset Y$ If $\{U_\alpha:\alpha\in I\}$ is a regularly open locally finite cover of X, then, $\{f(U_\alpha):\alpha\in I\}$ is an open star finite cover of Y.*

Proof. It is obvious that $\{f(U_\alpha):\alpha\in I\}$ is an open cover of Y. Thus we shall show it is star finite. Let $f(U_{\alpha 0})$ be any member of $\{f(U_\alpha):\alpha\in I\}$. Since $\{U_\alpha:\alpha\in I\}$ is locally finite, then, for each point $x\in f^{-1}[f(U_{\alpha 0})]$ there exists an open neighbourhood $G(x)$ of x in X and finite subfamily $I(x)$ of I such that $G(x)\cap U_\alpha=\varnothing$ for every $\alpha\in I\setminus I(x)$. The family $\{G(x):x\in f^{-1}[f(U_{\alpha 0})]\}$ is a cover of $f^{-1}[f(U_{\alpha 0})]$ by open sets of X. Since $f^{-1}[f(U_{\alpha 0})]$ is α-nearly compact there exists a finite number of points $x_1, x_2, \ldots, x_n\in f^{-1}[f(U_{\alpha 0})]$ such that

$$f^{-1}[f(U_{\alpha 0})]\subset\bigcup\{\alpha(G(x_i)):i=1, 2, \ldots, n\}.$$

Let $G=(\bigcup_{i=1}\overline{G(x_i)})^0\cdot$ G is a regularly open set of X containing $f^{-1}[f(U_{\alpha 0})]$. Since f is almost closed, there exists an open set V of Y containing $f(U_{\alpha 0})$ such that $f^{-1}(V)\subset G$. Thus we have

$$V\cap f(U_\alpha)=\varnothing \quad \text{for every} \quad \alpha\in I\setminus\bigcup_{i=1}^{n} I(x_i)$$

i. e. $f(U_{\alpha 0})\cap f(U_\alpha)=\varnothing$ for every $\alpha\in I\setminus\bigcup_{i=1}^{n} I(x_i)$.

This implies that $\{f(U_\alpha):\alpha\in I\}$ is star finite.

Theorem 1.5. *If f is almost closed, almost continuous and almost open mapping of a nearly paracompact space X onto a space Y such that $f^{-1}(G)$ is α-nearly compact for each proper open set $G\subset Y$, then Y is nearly strongly paracompact.*

Proof. Let $\mathcal{U}=\{U_\alpha:\alpha\in I\}$ be any regulary open covering of Y. Since f is almost continuous and almost open, $f^{-1}(\mathcal{U})=\{f^{-1}(U_\alpha):\alpha\in I\}$ is regular open cover of the space X. Since X is nearly paracompact, there exists a locally finite regularly open refinement $\mathcal{V}=\{V_\beta:\beta\in J\}$ of $f^{-1}(\mathcal{U})$. By the preceeding lemma $f(\mathcal{V})=\{f(V_\beta):\beta\in J\}$ is an open star finite refinement of $\mathcal{U}$. This implies that Y is nearly strongly paracompact.

Theorem 1.6. *If f is an almost continuous and almost open mapping of a nearly strongly paracompact space X onto a space Y, such that $f^{-1}(G)$ is α-nearly compact for every proper open set $G \subset Y$, then Y is nearly strongly paracompact.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be any regular open covering of Y. Then $f^{-1}(U_\alpha)$ is regular open for each α, since f is almost continuous and almost open mapping. Since X is nearly strongly paracompact, there exists a star finite, regular open refinement $\{V_\beta : \beta \in J\}$ of $\{f^{-1}(U_\alpha) : \alpha \in I\}$. Since f is almost open $\{f(V_\beta) : : \beta \in J\}$ is an open covering of Y. Since f is almost open, $f^{-1}(G)$ α-nearly compact for each proper open set $G \subset Y$ and $\{V_\beta : \beta \in J\}$ star finite, the family $\{f(V_\beta) : \beta \in J\}$ is a star finite. Thus $\{f(V_\beta) : \beta \in J)$ is a star finite, open refinement of $\{U_\alpha : \alpha \in I\}$, hence Y is nearly strongly paracompact.

Corollary 1.3. ([1]). *If f is an almost continuous and open mapping of a nearly strongly paracompact space X onto a space Y such that $f^{-1}(G)$ is compact for each proper open set $G \subset Y$, then Y is nearly strongly paracompact.*

2. Subsets and nearly strongly paracompact spaces

Definition 2.1. Let X be a topological space and A a subset of X. The set A is *α-nearly strongly paracompact* if and only if every regular open (in X) cover of A has an open (in X) star finite refinement which covers A. The subcet A is nearly strongly paracompact if and only if A is nearly strongly paracompact as a subspace.

Theorem 2.1. *Let X be an almost regular space and let A be any α-nearly strongly paracompact subset of X. Then, $\bar{A}$ is α-nearly strongly paracompact subset of X.*

Proof. Let A be any α-nearly strongly paracompact subset of Y. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any regular open (in X) cover of $\bar{A}$. For each $x \in \bar{A}$, there exists U_α such that $x \in U_\alpha$. Since X is almost regular, there exists a regularly open (in X) set V_x such that $x \in V_x \subset \bar{V}_x \subset U_\alpha$. Consider the regular open covering $\mathcal{V} = \{V_x : x \in A\}$ of A. By hypothesis, there exists a star finite family of open (in X) sets $\mathcal{W} = \{W_\beta : \beta \in J\}$ which refines $\mathcal{V}$ and covers A. The family $\{\alpha(W_\beta) : \beta \in J\}$ is star finite, regularly open, it refines $\{U_\alpha : \alpha \in I\}$ and covers A. Since X is almost regular for each $x \in A$, there exists a regularly open (in X) set V_x^* such that $x \in V_x^* \subset \bar{V}_x^* \in \alpha(W_\beta)$ for some $\beta \in J$. Since $\{V_x^* : x \in A\}$ is a regularly open covering of the α-nearly strongly paracompact subset A, there exists a star finite, open (in X) refinement $\{A_\lambda : \lambda \in \Lambda\}$ of $\{V_x^* : x \in A\}$ such that $A \subset \cup\{A_\lambda : \lambda \in \Lambda\}$. Then

$$\bar{A} \subset \overline{\cup\{A_\lambda : \lambda \in \Lambda\}} = \cup\{\bar{A}_\lambda : \lambda \in \Lambda\}.$$

$A_\lambda \subset V_{x^*}^*$, for some $x^* \in A$, i. e. $A_\lambda \subset V_{x^*}^* \subset \alpha(W_{\beta 0})$ for some $\beta_0 \in J$.

Thus, $\{\alpha(W_\beta) : \beta \in J\}$ is a star finite family of regularly open sets which refines the regular open family $\{U_\alpha : \alpha \in I\}$ such that $\bar{A} \subset \cup\{\alpha(W_\beta) : \beta \in J\}$. Hence $\bar{A}$ is an α-nearly strongly paracompact subset of the space X.

Theorem 2.2. *Let A be any subset of a space X. Then, A is α-nearly strongly paracompact subset of (X, τ) iff A is α-strongly paracompact subset of (X, τ^*).*

Proof. It is similar to the proof of Theorem 1.1.

Theorem 2.3. *Every τ^*-closed subset of a nearly strongly paracompact space is α-nearly strongly paracompact.*

Proof. Let A be any τ^*-closed subset of a nearly strongly paracompact space (X, τ). Since (X, τ^*) is strongly paracompact, then A is α-strongly paracompact subset of (X, τ^*). Hence by the preceding theorem A is α-nearly strongly paracompact subset of the space (X, τ).

Collary 2.1. *A clopen subset of a nearly strongly paracompact space is both α-nearly strongly paracompact as well as nearly strongly paracompact.*

Theorem 2.4. *A regulary closed subset of an α-nearly strongly paracompact set is itself α-nearly strongly paracompact.*

Proof. Let C be α-nearly strongly paracompact, B regularly closed in X and $B \subset C$. Let $\mathcal{U}$ be a regularly open (in X) cover of B. Then, $\mathcal{U} \cup \{X \setminus B\}$ is a regularly open cover of C, therefore there exists a star finite, open (in X) refinement $\mathcal{V}$ of $\mathcal{U}$, such that $C \subset \{X \setminus B\} \cup \{V : V \in \mathcal{V}\}$. It follows that $B \subset \cup \cup \{V : V \in \mathcal{V}\}$, so B is α-nearly strongly paracompact.

Theorem 2.5. *Let X be any topological space. A subset A of X is α-nearly strongly paracompact iff for each open covering $\mathcal{A}$ of A there exists a star finite family of open sets which refines it and the interiors of the closures of whose members cover the set A.*

Proof. To prove the „if" part, let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any regular open covering of A. Then, there exists a star finite family $\mathcal{V} = \{V_\beta : \beta \in J\}$ of open (in X) subsets of X such that V_β is contained in some U_α and $A \subset \cup \{\alpha(V_\beta) : \beta \in J\}$. Consider the family $\mathcal{V}^* = \{\alpha(V_\beta) : \beta \in J\}$. Clearly $\mathcal{V}^*$ is star finite open refinement of $\mathcal{U}$ and hence A is α-nearly strongly paracompact. To prove the „only if" part, let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be any open covering of A. Then, $\{\alpha(\overline{U_\lambda}) : \lambda \in \Lambda\}$ is a regular open covering of A. Since A is α-nearly strongly paracompact, there exists a star finite open (in X) refinement $\{H_\beta : \beta \in J\}$ of $\{\alpha(\overline{U_\lambda}) : \lambda \in \Lambda\}$ such that $H_\beta \subset \alpha(\overline{U_{\lambda(\beta)}})$ for some $\lambda(\beta) \in \Lambda$ and $A \subset \cup \{H_\beta : \beta \in J\}$. For each $\beta \in J$, let $M_\beta = H_\beta \setminus (\overline{U_{\lambda(\beta)}} \setminus U_{\lambda(\beta)})$. Since $H_\beta \subset \alpha(\overline{U_{\lambda(\beta)}}) \subset \overline{U_{\lambda(\beta)}}$, therefore $M_\beta = H_\beta \cap U_{\lambda(\beta)}$. Thus $\{M_\beta : \beta \in J\}$ is a star finite family of open sets which refines $\mathcal{U}$. We shall prove that $\cup \{\alpha(M_\beta) : \beta \in J\} \supset A$. Let $x \in A$, then $x \in H_\beta$ for some $\beta \in J$. Now $\alpha(M_\beta) = \alpha(H_\beta \cap U_{\lambda(\beta)} = \alpha(H_\beta \cap \overline{U_{\lambda(\beta)}}) = \alpha(H_\beta)$. Then $x \in \alpha(H_\beta) = \alpha(M_\beta)$. Hence $\{M_\beta : \beta \in J\}$ is a star finite family of open sets which refines $\mathcal{U}$ and the interiors of the closures of whose members cover A.

Theorem 2.7. *Let f be any star closed, almost continuous and almost open mapping of a space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$. Then, the image of α-nearly strongly paracompact subset of X is α-nearly strongly paracompact subset of Y.*

Proof. It is similar to the proof of Theorem 1.4.

Theorem 2.8. *Let f be any almost closed, almost continuous and almost open mapping of a space X onto a space Y such that $f^{-1}(G)$ is α-nearly compact for every proper open subset $G \subset Y$. Then, the image of an α-nearly paracompact subset of X is an α-nearly strongly paracompact subset of Y.*

Proof. It is similar to the proof of Theorem 1.5.

Theorem 2.9. *Let f be any almost continuous and almost open mapping of a space X onto a space Y, such that $f^{-1}(G)$ is α-nearly compact for every proper open set $G \subset Y$. Then, the image of an α-nearly strongly paracompact subset of X is an α-nearly strongly paracompact subset of Y.*

Proof. It is similar to the proof of Theorem 1.6.

3. Locally nearly strongly paracompact spaces

Definition 3.1. A topological space X is called *locally nearly strongly paracompact* iff each point has an open neighbourhood V, such that $\bar{V}$ is α-nearly strongly paracompact subset of X.

Lemma 3.1. *If B has an open neighbourhood U, such that $\bar{U}$ is α-nearly strongly paracompact, B also has a regularly open meighbourhoof V such that $\bar{V}$ is α-nearly strongly paracompact and $U \subset V \subset \bar{U}$.*

Proof. If $B \subset U$, and $\bar{U}$ α-nearly strongly paracompact then $B \subset \alpha(U) \subset \subset \overline{\alpha(U)} \subset \bar{U}$. Therefore $\alpha(U)$ is the desired neighbourhood.

Theorem 3.1. *The following conditions are equivalent in almost regular spaces:*

a) *The space X is locally nearly strongly paracompact,*

b) *For each x in X and each neighbourhood U of x, there exists an open set V such that $\bar{V}$ is α-nearly strongly paracompact, and $x \in V \subset \bar{V} \subset \alpha(U)$.*

c) *For each x in X and each regularly open neighbourhood U of x, there exists an open set V, such that $\bar{V}$ is α-nearly strongly paracompact and $x \in V \subset \bar{V} \subset U$.*

Proof. $(a) \to (b)$.

There exists an open set W with $x \in W \subset \bar{W}$ and $\bar{W}$ is α-nearly strongly paracompact. The set $\alpha(U \cap W)$ is regularly open and is contained in $\bar{W}$. There exists an open set V such that $x \in V \subset \bar{V} \subset \alpha(U \cap W) \subset \alpha(U)$. The set $\bar{V}$ is regularly closed contained in $\bar{W}$ so by theorem 2.4. $\bar{V}$ is α-nearly strongly paracompact. Hence the result.

$(b) \to (c)$ Clearly, by lemma 3.1.

$(c) \to (a)$ Obvious.

Theorem 3.2. *Let (X, τ) be a topological space. Then, (X, τ) is locally nearly strongly paracompact iff (X, τ^*) is locally strongly paracompact.*

Proof. Let (X, τ) be locally nearly strongly paracompact space, and x be any point of X. Then, by lemma 3.1. there exists a regularly open neighbourhood V of x, such that $\overline{V}$ is α-nearly strongly paracompact. Then, $\overline{V}$ is α-strongly paracompact set of (X, τ^*). Since V is regularly open, then, $\overline{V}$ is τ^*-closed set. Then V is τ^*-open neighbourhood of x, such that $\overline{V}_{\tau^*}=\overline{V}_\tau$ is α-strongly paracompact set of (X, τ^*). Hence (X, τ^*) is locally strongly paracompact.

Now, assume that (X, τ^*) is locally strongly paracompact and let $x \in X$. Then, there exists τ^*-open neighbourhood V of x such that $\overline{V}_{\tau^*}$ is strongly paracompact set of (X, τ^*). Then $\overline{V}_{\tau^*}$ is α-nearly strongly paracompact set of (X, τ). Since $\overline{V}_\tau \subset \overline{V}_{\tau^*}$ therefore $\overline{V}_\tau$ is α-nearly strongly paracompact subset of (X, τ).

Obviously, every nearly strongly paracompact space is locally nearly strongly paracompact. But a locally nearly strongly paracompact space may fail to be nearly strongly paracompact as is shown by the following example.

Example 3.1. Let Ω_0 be the set of all ordinal numbers less than the first uncountable ordinal Ω and let the topology be the order topology (the order topology has a subbase consisting of all sets of the form $\{x : x<a\}$ or $\{x : a<x\}$ for some a in Ω_0). Then Ω_0 is Hausdorff locally compact space. Therefore Ω_0 is regular space. Since every locally compact space is locally nearly strongly paracompact, Ω_0 is locally nearly strongly paracompact space. Ω_0 is well known not to be paracompact. Ω_0 is not nearly paracompact (Every regular nearly paracompact space is paracompact.) Therefore Ω_0 is not nearly strongly paracompact (Every nearly strongly paracompact is nearly paracompact).

Every locally nearly compact space is locally nearly strongly paracompact. The converse statement is not necessarily true. For our purpose, let X be any regular strongly paracompact space which is not locally compact. X is locally nearly strongly paracompact space which is not locally nearly compact (Every regular locally nearly compact space is locally compact.)

Obviously, every locally strongly paracompact space is locally nearly strongly paracompact. The converse statement is not necessarily true. The following example will serve the purpose.

Example 3.2. Let $X=\{a, a_i, a_{ij} : i, j=1, 2, \ldots\}$. Let each point a_{ij} be isolated. Let $\{U^k(a_j) : k=1, 2, \ldots\}$ be the fundamental system of neighbourhoods of a_i where $U^k(a_i)=\{a_i, a_{ij} \cdot j \geq k\}$ and let the fundamental system of neighbourhoods of a be $\{V^k(a) : k=1, 2, \ldots\}$ where $V^k(a)=\{a, a_{ij} : i \geq k, j \geq k\}$. Then X is a Hausdorff space which is not regular at a and hence X is not locally strongly paracompact (Every Hausdorff locally strongly paracompact space is regular). But X is locally nearly strongly paracompact (in fact nearly compact), for if $\mathcal{G}=\{G_\lambda : \lambda \in \Lambda\}$ be any open covering of X, then $a \in G_{\lambda(a)}$ for some $\lambda(a) \in \Lambda$. Denote by $G_{\lambda(i)}$ that $G_\lambda \in \mathcal{G}$ which contains a_i and $G_{\lambda(ij)}$ that which contains a_{ij}. Then, $V^m(a) \subset G_{\lambda(a)}$ for some m. Also $\alpha(V^m(a))=V^m(a) \cup \{a_m, a_{a+1}, \ldots\}$. Thus

$$\{G_{\lambda(a)}, G_{\lambda(ij)}, G_{\lambda(i)} : i=1, 2, \ldots, m-1, j=1, 2, \ldots, m-1\}$$

is a finite family of open sets which refines $\mathcal{G}$ and the interiors of the closures of whose members cover X. Hence X is nearly strongly paracompact.

Theorem 3.3. *The product of a locally nearly strongly paracompact and locally nearly compact space is locally nearly strongly paracompact.*

P r o o f. Let (x, y) be any point of $X \times Y$. Then $x \in X$ and $y \in Y$. Then, there exists a regular open neighbourhood A of x in X such that $\bar{A}$ is an α-nearly strongly paracompact subset of X. Also, there exists a regular open neighbourhood B of y in Y such that $\bar{B}$ is α-nearly compact subset of Y. Then, $A \times B$ is regularly open neighbourhood of (x, y) in $X \times Y$ such that $\overline{A \times B} = \bar{A} \times \bar{B}$ is α-nearly strongly paracompact.

T h e o r e m 3.4. *If f is star closed, almost continuous and almost open mapping of a locally nearly strongly paracompact space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each $y \in Y$, then Y is locally nearly strongly paracompact.*

P r o o f. For any point $y \in Y$, there exists a point $x \in X$ such that $f(x) = y$. There exists a regular open neighbourhood M of x such that $\bar{M}$ is α-nearly strongly paracompact subset of X. Since f is almost open, then $f(M)$ is open set in Y containing y, such that $\overline{f(M)} \subset f(\bar{M})$, ($f$ is star closed mapping. therefore $f(\bar{M})$ is closed subset of Y) and $f(\bar{M})$ is α-nearly strongly paracompact subset of Y. Therefore, since $\overline{f(M)}$ is regularly closed subset of $f(\bar{M})$, $\overline{f(M)}$ is α-nearly strongly paracompact subset of Y. Hence the result.

T h e o r e m 3.5. *Let f be any almost closed, almost continuous and almost open mapping of a locally nearly strongly paracompact space X onto a space Y, such that $f^{-1}(G)$ is α-nearly compact subset of X for every proper open subset $G \subset Y$, then Y is locally nearly strongly paracompact.*

P r o o f. It is similar to the proof of preceeding theorem.

T h e o r e m 3.6. *Let E be an α-nearly strongly paracompact subset of a locally nearly strongly paracompact, almost regular space X, and let $\{F_j : F_j^0$ is not empty $j \in J\}$ be a star finite family of closed α-nearly strongly paracompact sets which contain E in the union of their interiors. Then there exists a star finite family $\{A_j : j \in J\}$ of closed α-nearly strongly paracompact sets which contain E in the union of their interiors and $A_j \subset F_j^0$ for each $j \in J$.*

P r o o f. For each $x \in E$, $x \in F_j^0$ for some $j \in J$. Since X is almost regular, there exists regular open neighbourhood G_x of x such that $x \in G_x \subset \bar{G}_x \subset F_j^0$. Since $\bar{G}_x \subset F_j$, therefore $\bar{G}_x$ is α-nearly strongly paracompact subset of X. By lemma 3.1. there exists a regular open neighbourhood V_x of x such that $x \in V_x \subset \bar{V}_x \subset G_x \subset F_j^0$ and $\bar{V}_x$ is α-nearly strongly paracompact subset of X. Since $\mathcal{V} = \{V_x : x \in E\}$ is regular open cover of E, there exists a regularly open (in X) star finite refinement $\mathcal{W}$ of $\mathcal{V}$ which covers E. The family $\{\bar{W} : W \in \mathcal{W}\}$ is a closed star finite cover of E, and for every $W \in \mathcal{W}$, $\bar{W}$ is α-nearly strongly paracompact as a regularly closed subset of $\bar{V}_x$ for some $x \in E$.

Now observe that every set $\bar{W}$ is contained in F_j^0 for some F_j. Let $A_j = \bigcup\{\bar{W} : \bar{W} \subset F_j^0\}$. Clearly A_j are closed and since $\bigcup\{W : \bar{W} \subset F_j^0\}$ is contained in A_j^0 we have E contained in $\bigcup\{A_j^0 : j \in J\}$. Clearly $\{A_j : j \in J\}$ is star finite and A_j contained in F_j yields A_j is α-nearly strongly paracompact.

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