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SOME PROPERTIES OF FULLY SUBMITTED PROCESSES

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1. Let $X=\{X(t),\ 0\leq t\leq 1\}$ be a random process of second order, i.e. such that $\|X(t)\|^2=E|X(t)|^2<\infty$ for all $0\leq t\leq 1$. We denote by $H(X;t)$ $(H(X;t-0))$, $0\leq t\leq 1$, the Hilber. space obtained as a closure (in quadratic mean) of the linear manifold gen^rated by the elements $X(s)$, $s\leq t$ $(s<t)$: $H(X;t)=\overline{\mathcal{L}}\{X(s), s\leq t\}$ $(H X;t-0)=\overline{\mathcal{L}}\{X(s),\ s<t\})$; put $H(X)=H(X;1)$. We suppose X to be nondeterministic, i.e. $H(X;0)=0$. The orthogonal complement of $H(X;t)$ with respect to $H(X)$ we denote by $H^{\perp}(X;t)$: $H^{\perp}(X;t)=H(X)\ominus H(X;t)$. For the process X (as well as for the other processes) the following equality will be valid: $H(X;t-0)=H(X;t)$.

The projection operator from $H(X)$ onto $H(X;t)$ we denote by $E_X(t)$; it is ea y to see that the family $E_X=\{E_X(t),\ 0\leq t\leq 1\}$ represents a resolution of the identity of the sp ce $H(X)$, [1, 3]. Every element $z\in H(X)$ generates the measure m_z induced by $F_z(t)=\|E_X(t)z\|^2$, $0\leq t\leq 1$. We introduce in the ordinary way the equivalence relation in the set of all measures generated by the elemets from $H(X)$: two measures are equivalent if and only if they are mutually absolutely continuous. The spectral type of the element z is the equivalence class of the measure m_z.

The Hilbert space $\mathcal{M}_z$ spanned by the elements $E_X(t)z$, $0\leq t\leq 1$, for arbitrary $z\in H(X)$, we call the cyclic space with respect to E_X, generated by z: $\mathcal{M}_z=\overline{\mathcal{L}}\{E_X(t)z,\ 0\leq t\leq 1\}$; the spectral type of $\mathcal{M}_z$ is the spectral type of the element z.

We say that the arbitrary subspace $\mathcal{M}$ of $H(X)$ is invariant with respect to E_X if $E_X(t)\mathcal{M}\subset\mathcal{M}$ for all t, and that $\mathcal{M}$ reduces E_X if both $\mathcal{M}$ and $H(X)\ominus\mathcal{M}$ are invarian. with respec. to E_X. It is easy to see that, for any $z\in H(X)$, cyclic sub pace $\mathcal{M}_z$ reduces E_X.

If a random process Y of second order is such that the relation

$$H(Y;t)\subset H(X;t) \text{ for all } t$$

holds, we say that Y is submitted to X; if together with the preceding relation,

$$H^{\perp}(Y;t)\subset H^{\perp}(X;t) \text{ for all } t$$

holds, we say that Y is fully submit ed to X. It can be shown [2] that Y is fully submitted to X if and only if for all t the subspace $H(Y;t)$ reduces the resolution of the identity E_X of $H(X)$.

In this paper we shall find, under the assumption that Y is submitted to X, conditions for the process Y^*, submitted to Y and fully submitted to X, to exist. The method we use is constructive, so that, in case that Y^* exists, we can construct subspaces $H(Y^*; t)$. Y^* can be considered as the best estimation of the process Y by a process which is submitted to Y and fully submitted to X, in the sense that, if Y^{**} is some other process submitted to Y and fully submitted to X, then it will be $H(Y^{**}; t) \subset H(Y^*; t)$ for all t.

2. Definition 1. We say that the element $x \in H(Y)$ is *s-regular* if $x \in H^{\perp}(Y; s)$ and $x \in H^{\perp}(X; s)$.

Obviously, every element from $H(Y)$ is 0-regular, so that the only interesting case is when, for $x \in H(Y)$, there exists an $s > 0$ such that x is s-regular.

The number s_x we define by

$$s_x = \sup\{s : x \in H^{\perp}(Y; s)\}. \tag{1}$$

Lema 1. The relation $s_x \in \{s : x \in H^{\perp}(Y; s)\}$ is valid for every $x \in H(Y)$.

Proof: From (1) it follows that $x \perp H(Y; s)$ for all $s < s_x$, i.e. that $x \perp H(Y; s_x - 0)$. But, as $H(Y; t-0) = H(Y; t)$ for all t, we have $x \perp H(Y; s_x)$, which is equivalent with the assertion of the lemma.

Definition 2. An element $x \in H(Y)$ is *regular*, if it is s_x-regular, where s_x is defined by (1). The number s_x is the *regularity level* of the element x.

Let us suppose that there exists at least one regular element in $H(Y)$; later, we shall show by an example that the set of regular elements can be empty.

Definition 3. We say that the element $x \in H(Y)$ is *strictly regular* if the elements $E_Y(t)x$ and $x - E_Y(t)x$ are regular for all t.

Obviously, every strictly regular element is regular. By the following example we shall show that the inverse does not hold, i.e. that from the fact that x is regular it does not follow that $E_Y(t)x$ and $x - E_Y(t)x$ are regular for all t.

Example 1. Let $x = x_1 + x_2$, $s_{x_1} < s_{x_2}$, x_1 is regular and $x_1 \in H(Y; s_{x_2})$. We suppose that x_2 is s_{x_1}-regular, but not regular, which is possible, because of $s_{x_1} < s_{x_2}$. Then, $x - E_Y(s_{x_2})x = x_2$ is not regular, so that x is not strictly regular.

For a (strictly) regular element x the relation $t \leq s_x$ is equivalent to $x \perp H(X; t)$. For every $x \in H(Y; t)$, (strictly) regular or not, it follows $s_x < t$. Therefore $s_t < t$, for every t, where

$$s_t = \sup\{s : Y(t) \in H^{\perp}(Y; s)\}.$$

Theorem 1. *Element $x \in H(Y)$ is strictly regular if and only if*

$$E_Y(t)x = E_X(t)x \text{ for all } t. \tag{2}$$

Proof: Let x be strictly regular, and t_0 an arbitrary number from $(s_x, s_x'']$, where

$$s_x''=\inf\{s: x\in H(Y;s)\}.$$

Then, obviously, (2) is valid for $t\leq s_x$ and $t>s_x''$. Let

$$x_1(t)=E_Y(t)x, \quad x_2(t)=x-E_Y(t)x.$$

It is easy to see that the equality (2) is equivalent with

(3) $$P_{H(X;t)\ominus H(Y;t)}x=0 \text{ for all } t.$$

Let us put

(4) $$P_{H(X;t)\ominus H(Y;t)}x=x_t,$$

and show that $x_t=0$ for any t. As $x=x_1(t_0)+x_2(t_0)$ for any t, it follows from (4) that

$$x_t=P_{H(X;t_0)\ominus H(Y;t_0)}x_2(t_0).$$

But, as $x_2(t_0)$ is regular and $x_2(t_0)\perp H(Y;t_0)$, we have $x_2(t_0)\perp H(X;t_0)$ and that implies $x_t=0$, which, because of (4), means that (3), i.e. (2) is valid.

Let us suppose that (2) holds. It follows that

$$x_1(t)=E_X(t)x, \quad x_2(t)=x-E_X(t)t,$$

so that we immediately conclude that $x_1(t)$ and $x_2(t)$ are regular for any t, i.e. x is strictly regular. Really, if $x_1(t)\perp H(Y;s)$ for some $s<t$, that means that

$$0=E_Y(s)x_1(t)=E_Y(s)x=E_X(s)x=E_X(s)E_X(t)x=$$
$$=E_X(s)E_Y(t)x=E_X(s)x_1(t),$$

that is $x_1(t)\perp H(X;s)$. In the same way, if $x_2(t)\perp H(Y;s)$ for some $s\geq t$, then

$$0=E_Y(s)x_2(t)=E_X(s)x-E_X(t)x=E_X(s)(x-E_X(t)x)=$$
$$=E_X(s)(x-E_Y(t)x)=E_X(s)x_2(t),$$

which is equivalent with $x_2(t)\perp H(X;s)$.

Consequence 1. If x is striotly regular, then the elements $E_Y(t)x$ and $x-E_Y(t)x$ are strictly regular for all t.

Lemma 2. A finite linear combination of strictly regular elements is strictly regular.

Proof. It is enough to prove that the assertion holds for the sum of two strictly regular elements x and y. Let s_x and s_y be the regularity levels of x and y, respectively; suppose that $s_x\leq s_y$. Put $z=x+y$ and

$$z_1(t)=E_Y(t)z=x_1(t)+y_1(t); \quad z_2(t)=z-E_Y(t)z=x_2(t)+y_2(t).$$

First, we shall show that z is regular. Obviously, for the regularity level of z we have $s_z\geq s_x$. Only when $s_x=s_y$, it could happen that $s_z>s_x$; when $s_x<s_y$ we have always that $s_z=s_x$.

Let us suppose that $s_z \geq s_x$. Then it must be $x_1(s_z) = -y_1(s_z)$, from which we have that $z = x_2(s_z) + y_2(s_z)$. From the strictly regularity of x and y, and from Theorem 1 it follows that $z \perp H(X; s_z)$, that is z is regular. The regularity of the elements $z_1(t)$ and $z_2(t)$ follows immediately from the Consequence 1 and the above proof of the regularity of z. Hence, by Definition 3, z is strictly regular.

Lema 3. If $\{x_n\}_1^\infty$ is a convergent sequence of strictly regular elements from $H(Y)$ and $\underset{n\to\infty}{l.\,i.\,m.}\, x_n = x$, then x is strictly regular.

Proof: We shall show first a) that $E_Y(t)x$ is regular, and then b) that $x - E_Y(t)x$ is regular.

a) $E_Y(t)x$ is regular if, together with all s for which $E_Y(t)x \in H^\perp(Y; s)$, we have that $E_Y(t)x \in H^\perp(X; s)$. Let us suppose that for some $s\,(s<t)$ the relation $E_Y(t)x \in H^\perp(Y; s)$, i.e. $E_Y(t)x \perp H(Y; s)$, holds. Then, by reason of strictly regularity of the elements x_n, $n = 1, 2, \ldots$, and because of Theorem 1, we have

$$0 = E_Y(s)\,E_Y(t)\,x = \underset{n\to\infty}{l.i.m.}\, E_Y(s)\,x_n = \underset{n\to\infty}{l.\,i.\,m.}\, E_X(s)\,x_n = E_X(s)\,E_Y(t)\,x.$$

Therefore, element $E_Y(t)x$ is regular for all t. It remains to prove b) that $x - E_Y(t)x = x_2(t)$ is regular. If $x_2(t) \perp H(Y; s)$ for some $s\,(s \geq t)$, then, by the same reasons as above, we have

$$0 = E_Y(s)\,(x - E_Y(t)\,x) = \underset{n\to\infty}{l.\,i.\,m.}\, E_Y(s)\,x_n - \underset{n\to\infty}{l.\,i.\,m.}\, E_Y(t)\,x_n =$$

$$= \underset{n\to\infty}{l.\,i.\,m.}\, E_X(s)\,x_n - \underset{n\to\infty}{l.\,i.\,m.}\, E_X(t)\,x_n = E_X(s)\,(x - E_Y(t)\,x).$$

The natural question is: what could be said about the regularity level of the limit of a sequence of (strictly) regular elements? Let $s = \overline{\lim_{n\to\infty}}\, s_{x_n}$. From the sequence $\{x_n\}_1^\infty$, $\underset{n\to\infty}{l.\,i.\,m.}\, x_n = x$, we can choose a subsequence $\{x_{n_k}\}_1^\infty$ such that $\lim_{k\to\infty} s_{x_{n_k}} = s$. As elements x_{n_k} are (strictly) regular and $x_{n_k} \perp H(Y; s_{n_{n_k}})$, it follows that

$$x \perp H(Y; \lim_{k\to\infty} s_{x_{n_k}}) \text{ and } x \perp H(X; \lim_{k\to\infty} s_{x_{n_k}}),$$

i.e. x is s-regular. Therefore, we have

$$s_{l.\,i.\,m.\,x_n} \geq \overline{\lim_{n\to\infty}}\, s_{x_n}. \tag{5}$$

The following example shows that in (5) we can have a strict inequality.

Example 2. Let y_1 and y_2 be two arbitrarily chosen strictly regular elements, such that $s_{y_1} < s_{y_2}$. Let $x_n = a_n y_1 + y_2$, $n = 1, 2, \ldots$, where $\{a_n\}_1^\infty$ is a sequence of real numbers converging to zero. It is easy to see that $\underset{n\to\infty}{l.\,i.\,m.}\, x_n = s_{y_2}$ which means that $s_{l.\,i.\,m.\,x_n} = s_{y_2}$. However, as

$$s_{x_n} = \{\min s_{y_1}, s_{y_2}\} = s_{y_1},$$

it follows that

$$s_{l.\,i.\,m.\,x_n} = s_{y_2} > s_{y_1} = \overline{\lim_{n\to\infty}}\, s_{x_n} = \lim_{n\to\infty} s_{x_n}.$$

3. Let us denote bu H^* the set of all strictly regular elements from $H(Y)$. It follows from Lemmas 2 and 3 that H^* is a Hilbert space. In order to define a process Y^*, fully submitted to X and submitted to Y, it is enough to determine the family of subspaces H_t^*, $0 \leq t \leq 1$, such that H_t^* reduces the resolution of the identity $E_X = \{E_X(s),\ 0 \leq s \leq 1\}$ of X and $H_t^* \subset H(Y; t)$, $0 \leq t \leq 1$, [2, 3]. Really, if H_t^*, $0 \leq t \leq 1$, is an arbitrary nondecreasing family of subspaces, a process Y^*, such that $H(Y^*; t) = H_t^*$, $0 \leq t \leq 1$, always exists and has required properties, [1].

Let the family H_t^*, $0 \leq t \leq 1$, be defined bu the equality

$$H_t^* = E_Y(t)\, H^*, \quad 0 \leq t \leq 1.$$

Denote by Y^* the process which satisfeis the equality

$$H(Y^*; t) = H_t^*, \quad 0 \leq t \leq 1.$$

Obviously, Y^* is submitted to Y, and then to X, too; let us show that it is fully submitted to X. First of all, we have that $E_Y(t)\, x \in H^*$, for any $x \in H^*$. Really, since x is strictly regular, $E_Y(t)\, x$ is also strictly regular, and therefore $E_Y(t)\, x \in H^*$. Hence $H_t^* \subset H^*$. It is easy to see that any element $x \in H^*$, which is orthogonal to H_t^* for some t, will be also orthogonal to $H(Y; t)$; really, if $E_Y(t)\, x = x_1 \neq 0$, then, by reason of $H_t^* \subset H^*$, x_1 belongs to H_t^*, which contradicts our assumption that $x \in H^* \ominus H_t^*$. Thus, the following theorem is proved.

T h e o r e m 2. *Process Y^*, for which the equality*

$$H(Y^*; t) = E_Y(t)\, H^*, \quad 0 \leq t \leq 1,$$

is valid, is submitted to Y and fully submitted to X.

R e m a r k. It is easy to see that the following equalities are valid:

$$H_t^* = \{y \in H^*;\ y \in H(Y; t)\}$$

$$H_t^{*\perp} = \{y \in H^*;\ y \in H^\perp(Y; t)\}.$$

The next theorem gives one interesting and not so obvious property of Y^*.

T h e o r e m 3. *Let Y be an arbitrary process, submitted to X. If Y^* is a process submitted to Y and fully submitted to X, then Y^* is also fully submitted to Y.*

P r o o f. As Y^* is fully submitted to X, the space $H(Y^*)$ could be represented as orthogonal sum of the subspaces of $H(X)$, cyclic with respect to E_X. If we denote the generating elements of those cyclic subspaces by z_ν, $\nu \in \mathcal{N}$, and the corresponding subspaces by $\mathcal{M}_{z_\nu}$, $\nu \in \mathcal{N}$, then we shall have

$$H(Y^*) = \sum_\nu \oplus \mathcal{M}_{z_\nu} \text{ and } H(Y^*; t) = \sum_\nu \oplus E_X(t)\, \mathcal{M}_{z_\nu}$$

(if dim $H(Y^*)=\aleph_1$, then card $\mathcal{N}=\aleph_1$, and the last sum would be formal). If, for every $\nu\in\mathcal{N}$, we define the process Z_ν by

$$Z_\nu(t)=E_X(t)\,z_\nu,\quad 0\le t\le 1,$$

then Z_ν would be the process with orothogonal increments and the equality

$$H(Z_\nu;\,t)=E_X(t)\,\mathcal{M}_{z_\nu},\quad 0\le t\le 1,$$

will hold for any $\nu\in\mathcal{N}$; it is clear that every process Z_ν is fully submitted to X.

It is easy to see that

$$H^\perp(Z_\nu;\,t)=\overline{\mathcal{L}}\{Z_\nu(s)-Z_\nu(t),\quad t<s\le 1\},\quad 0\le t\le 1.$$

But, as $H(Y^*;\,t)\subset H(Y;\,t)$ for every t, we have $E_X(t)\,z_\nu=E_Y(t)\,z_\nu$ for every t, and then

$$H^\perp(Z_\nu;\,t)=\overline{\mathcal{L}}\{(E_Y(s)-E_Y(t))\,z_\nu,\quad t<s\le 1),\quad 0\le t\le 1.$$

As $(E_Y(s)-E_Y(t))\,z_\nu\in H^\perp(Y;\,t)$ for all $s>t$, then $H^\perp(Z_\nu;\,t)\subset H^\perp(Y;\,t)$ for every t. Therefore, the process Z_ν is, for every ν, fully submitted to Y, which implies that Y^* is fully submitted to Y, too.

By the following example we shall show that Y^* does not always exist.

Example 3. Let ω be a Wiener process defined on [0; 1], and let

$$X(t)=\omega(t),\quad Y(t)=\omega(t^2),\quad 0\le t\le 1.$$

It is clear that the equality $H(X)=H(Y)$ holds, but, for any $t<1$, we have

$$H(Y;\,t)=\overline{\mathcal{L}}\{\omega(u^2),\quad u\le t\}=\overline{\mathcal{L}}\{\omega(u),\quad u\le t^2\}=H(X;\,t^2)\subset H(X;\,t),$$

which means that Y is submitted to X. Let us show that Y is not fully submitted to X. From the assumption that Y is fully submitted to X it follows that any element $x\in H(Y)$, which is orthogonal to $H(Y;\,t)$ for some t, will be also orthogonal to $H(X;\,t)$; it is easy to see that such an element x, bacause of $H(Y;\,t)=H(X;\,t^2)$, can have the following form:

$$x=\int_{t^2}^{1} f(u)\,d\omega(u),$$

where f is a function from $L_2(dt)$, not identicaly equal to zero on $(t^2,\,t)$. But, such x is not orthogonal to $H(X;\,t)$, since it is not orthogonal, for example, to the element $\int_{t^2}^{1} f(u)\,d\omega(u)$, which belongs to $H(X;\,t)$. Thus, we proved that Y is not fully submitted to X. Let us show that $H(Y)$ does not contain any regular element. If such an element z exists (let s_z be its regularity level), then, as follows from the previous discussion, z could de represented in the form

$$z=\int_{s_z^2}^{1} f(u)\,d\omega(u),$$

where $f(u) \neq 0$ for almost all $u \in (s_z^2, s_z^2 + h)$, for some $h > 0$ (because, othewise, s_z could not be the regularity level for z). But we have already shown that such element is not orthogonal to $H(X; s_z)$, which means that it is not regular. Thus, we showed that $H(Y)$ does not contain any regular element and the consequence of this fact is that the process Y^*, submitted to Y and fully submitted to X, does not exist.

It is clear that the necessary and sufficient condition for the process Y^* to exist is the existence of least one regular element. Now, we shall give the necessary and sufficient conditions for the existence of Y^* in terms of the resolution of the identity of X.

Theorem 4. *The process Y^* exists if and only if there exists at least one element $z \in H(Y)$, such that*

$$E_X(t)\, z \in H(Y; t) \text{ for all } t. \tag{6}$$

Proof. Let us first suppose that Y^* exists; from this assumption it follows that in $H(Y)$ there exists at least one strictly regular element z. For this element z, by reason of Theorem 1, the equality $E_X(t)\, z = E_Y(t)\, z$ holds for any t, which implies that (6) holds.

Let us suppose now that there exists an element z from $H(Y)$ such that (6) is valid. We define Y^* by

$$Y^*(t) = E_X(t)\, z, \quad 0 \leq t \leq 1;$$

from this definition it is clear that Y^* is fully submitted to X, and from (6) it follows that it is submitted to Y.

From Theorems 3 and 4 it follows that a necessary condition for the existence of Y^* can be expressed in terms of spectral types of X and Y.

Theorem 5. *Let ρ_X and ρ_Y be maximal spectral types*[1] *of the processes X and Y, respectively. The necessary condition for the existence of Y^* is that inf $\{\rho_X, \rho_Y\} \neq 0$*[3].

Proof. If Y^* exists, then, according to Theorem 3, it is fully submitted to both X and Y. For an arbitrary z from $H(Y^*)$, we have, according to Theorem 4, $E_X(t)\, z = E_Y(t)\, z$, $0 \leq t \leq 1$, which means that z generates the same spectral type ρ_z with respect to both resolutions of the identity. Thus, we have $\rho_z \neq 0$, $\rho_z < \rho_X$ and $\rho_z < \rho_Y$, from where we have that $\inf\{\rho_X, \rho_Y\} > \rho_z \neq 0$, and our assertion is proved.

The following example will show that $\inf\{\rho_X, \rho_Y\} \neq 0$ is not a sufficient condition for Y^* to exist.

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[1] We say that ρ_X is the maximal spetral type of X if any element $z \in H(X)$ induces the spectral type ρ_z which is subordinated[2] to ρ_X.

[2] We say that the spectral type ρ is subordinated to spectral type μ, and we write $\rho < \mu$, if any measure of the spectral type ρ is absolutely continuous with respect to any measure of the spectral type μ.

[3] By $\inf\{\rho, \mu\}$ we denote the greatest spectral type which is subordinated to ρ and μ.

Example 4. Let the processes X and Y be defined as in Example 3. Both processes are Wiener, so that they have a unit multiplicities and the spectral types ρ_X and ρ_Y are equal to the ordinary Lebesgue measure. Therefore, $\inf \{\rho_X, \rho_Y\} \neq 0$, although Y^* does not exist, as it is shown in Example 3.

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