

## SOME PROPERTIES OF FULLY SUBMITTED PROCESSES

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1. Let  $X = \{X(t), 0 \leq t \leq 1\}$  be a random process of second order, i.e. such that  $\|X(t)\|^2 = E|X(t)|^2 < \infty$  for all  $0 \leq t \leq 1$ . We denote by  $H(X; t)$  ( $H(X; t-0)$ ),  $0 \leq t \leq 1$ , the Hilbert space obtained as a closure (in quadratic mean) of the linear manifold generated by the elements  $X(s)$ ,  $s \leq t$  ( $s < t$ ):  $H(X; t) = \overline{\mathcal{L}}\{X(s), s \leq t\}$  ( $H(X; t-0) = \overline{\mathcal{L}}\{X(s), s < t\}$ ); put  $H(X) = H(X; 1)$ . We suppose  $X$  to be nondeterministic, i.e.  $H(X; 0) = 0$ . The orthogonal complement of  $H(X; t)$  with respect to  $H(X)$  we denote by  $H^\perp(X; t)$ :  $H^\perp(X; t) = H(X) \ominus H(X; t)$ . For the process  $X$  (as well as for the other processes) the following equality will be valid:  $H(X; t-0) = H(X; t)$ .

The projection operator from  $H(X)$  onto  $H(X; t)$  we denote by  $E_X(t)$ ; it is easy to see that the family  $E_X = \{E_X(t), 0 \leq t \leq 1\}$  represents a resolution of the identity of the space  $H(X)$ , [1, 3]. Every element  $z \in H(X)$  generates the measure  $m_z$  induced by  $F_z(t) = \|E_X(t)z\|^2$ ,  $0 \leq t \leq 1$ . We introduce in the ordinary way the equivalence relation in the set of all measures generated by the elements from  $H(X)$ : two measures are equivalent if and only if they are mutually absolutely continuous. The spectral type of the element  $z$  is the equivalence class of the measure  $m_z$ .

The Hilbert space  $\mathcal{M}_z$  spanned by the elements  $E_X(t)z$ ,  $0 \leq t \leq 1$ , for arbitrary  $z \in H(X)$ , we call the cyclic space with respect to  $E_X$ , generated by  $z$ :  $\mathcal{M}_z = \overline{\mathcal{L}}\{E_X(t)z, 0 \leq t \leq 1\}$ ; the spectral type of  $\mathcal{M}_z$  is the spectral type of the element  $z$ .

We say that the arbitrary subspace  $\mathcal{M}$  of  $H(X)$  is invariant with respect to  $E_X$  if  $E_X(t)\mathcal{M} \subset \mathcal{M}$  for all  $t$ , and that  $\mathcal{M}$  reduces  $E_X$  if both  $\mathcal{M}$  and  $H(X) \ominus \mathcal{M}$  are invariant with respect to  $E_X$ . It is easy to see that, for any  $z \in H(X)$ , cyclic subspace  $\mathcal{M}_z$  reduces  $E_X$ .

If a random process  $Y$  of second order is such that the relation

$$H(Y; t) \subset H(X; t) \text{ for all } t$$

holds, we say that  $Y$  is submitted to  $X$ ; if together with the preceding relation,

$$H^\perp(Y; t) \subset H^\perp(X; t) \text{ for all } t$$

holds, we say that  $Y$  is fully submitted to  $X$ . It can be shown [2] that  $Y$  is fully submitted to  $X$  if and only if for all  $t$  the subspace  $H(Y; t)$  reduces the resolution of the identity  $E_X$  of  $H(X)$ .

In this paper we shall find, under the assumption that  $Y$  is submitted to  $X$ , conditions for the process  $Y^*$ , submitted to  $Y$  and fully submitted to  $X$ , to exist. The method we use is constructive, so that, in case that  $Y^*$  exists, we can construct subspaces  $H(Y^*; t)$ .  $Y^*$  can be considered as the best estimation of the process  $Y$  by a process which is submitted to  $Y$  and fully submitted to  $X$ , in the sense that, if  $Y^{**}$  is some other process submitted to  $Y$  and fully submitted to  $X$ , then it will be  $H(Y^{**}; t) \subset H(Y^*; t)$  for all  $t$ .

**2. Definition 1.** We say that the element  $x \in H(Y)$  is *s-regular* if  $x \in H^\perp(Y; s)$  and  $x \in H^\perp(X; s)$ .

Obviously, every element from  $H(Y)$  is 0-regular, so that the only interesting case is when, for  $x \in H(Y)$ , there exists an  $s > 0$  such that  $x$  is *s-regular*.

The number  $s_x$  we define by

$$(1) \quad s_x = \sup \{s : x \in H^\perp(Y; s)\}.$$

**Lemma 1.** The relation  $s_x \in \{s : x \in H^\perp(Y; s)\}$  is valid for every  $x \in H(Y)$ .

**Proof:** From (1) it follows that  $x \perp H(Y; s)$  for all  $s < s_x$ , i.e. that  $x \perp H(Y; s_x - 0)$ . But, as  $H(Y; t - 0) = H(Y; t)$  for all  $t$ , we have  $x \perp H(Y; s_x)$ , which is equivalent with the assertion of the lemma.

**Definition 2.** An element  $x \in H(Y)$  is *regular*, if it is  $s_x$ -regular, where  $s_x$  is defined by (1). The number  $s_x$  is the *regularity level* of the element  $x$ .

Let us suppose that there exists at least one regular element in  $H(Y)$ ; later, we shall show by an example that the set of regular elements can be empty.

**Definition 3.** We say that the element  $x \in H(Y)$  is *strictly regular* if the elements  $E_Y(t)x$  and  $x - E_Y(t)x$  are regular for all  $t$ .

Obviously, every strictly regular element is regular. By the following example we shall show that the inverse does not hold, i.e. that from the fact that  $x$  is regular it does not follow that  $E_Y(t)x$  and  $x - E_Y(t)x$  are regular for all  $t$ .

**Example 1.** Let  $x = x_1 + x_2$ ,  $s_{x_1} < s_{x_2}$ ,  $x_1$  is regular and  $x_1 \in H(Y; s_{x_2})$ . We suppose that  $x_2$  is  $s_{x_1}$ -regular, but not regular, which is possible, because of  $s_{x_1} < s_{x_2}$ . Then,  $x - E_Y(s_{x_2})x = x_2$  is not regular, so that  $x$  is not strictly regular.

For a (strictly) regular element  $x$  the relation  $t \leq s_x$  is equivalent to  $x \perp H(X; t)$ . For every  $x \in H(Y; t)$ , (strictly) regular or not, it follows  $s_x < t$ . Therefore  $s_t < t$ , for every  $t$ , where

$$s_t = \sup \{s : Y(t) \in H^\perp(Y; s)\}.$$

**Theorem 1.** Element  $x \in H(Y)$  is strictly regular if and only if

$$(2) \quad E_Y(t)x = E_X(t)x \text{ for all } t.$$

**Proof:** Let  $x$  be strictly regular, and  $t_0$  an arbitrary number from  $(s_x, s_x'']$ , where

$$s_x'' = \inf \{s : x \in H(Y; s)\}.$$

Then, obviously, (2) is valid for  $t \leq s_x$  and  $t > s_x''$ . Let

$$x_1(t) = E_Y(t)x, \quad x_2(t) = x - E_Y(t)x.$$

It is easy to see that the equality (2) is equivalent with

$$(3) \quad P_{H(X; t) \ominus H(Y; t)} x = 0 \text{ for all } t.$$

Let us put

$$(4) \quad P_{H(X; t) \ominus H(Y; t)} x = x_t,$$

and show that  $x_t = 0$  for any  $t$ . As  $x = x_1(t_0) + x_2(t_0)$  for any  $t$ , it follows from (4) that

$$x_t = P_{H(X; t_0) \ominus H(Y; t_0)} x_2(t_0).$$

But, as  $x_2(t_0)$  is regular and  $x_2(t_0) \perp H(Y; t_0)$ , we have  $x_2(t_0) \perp H(X; t_0)$  and that implies  $x_t = 0$ , which, because of (4), means that (3), i.e. (2) is valid.

Let us suppose that (2) holds. It follows that

$$x_1(t) = E_X(t)x, \quad x_2(t) = x - E_X(t)x,$$

so that we immediately conclude that  $x_1(t)$  and  $x_2(t)$  are regular for any  $t$ , i.e.  $x$  is strictly regular. Really, if  $x_1(t) \perp H(Y; s)$  for some  $s < t$ , that means that

$$\begin{aligned} 0 &= E_Y(s)x_1(t) = E_Y(s)x = E_X(s)x = E_X(s)E_X(t)x = \\ &= E_X(s)E_Y(t)x = E_X(s)x_1(t), \end{aligned}$$

that is  $x_1(t) \perp H(X; s)$ . In the same way, if  $x_2(t) \perp H(Y; s)$  for some  $s \geq t$ , then

$$\begin{aligned} 0 &= E_Y(s)x_2(t) = E_X(s)x - E_X(t)x = E_X(s)(x - E_X(t)x) = \\ &= E_X(s)(x - E_Y(t)x) = E_X(s)x_2(t), \end{aligned}$$

which is equivalent with  $x_2(t) \perp H(X; s)$ .

**Consequence 1.** If  $x$  is strictly regular, then the elements  $E_Y(t)x$  and  $x - E_Y(t)x$  are strictly regular for all  $t$ .

**Lemma 2.** A finite linear combination of strictly regular elements is strictly regular.

**Proof.** It is enough to prove that the assertion holds for the sum of two strictly regular elements  $x$  and  $y$ . Let  $s_x$  and  $s_y$  be the regularity levels of  $x$  and  $y$ , respectively; suppose that  $s_x \leq s_y$ . Put  $z = x + y$  and

$$z_1(t) = E_Y(t)z = x_1(t) + y_1(t); \quad z_2(t) = z - E_Y(t)z = x_2(t) + y_2(t).$$

First, we shall show that  $z$  is regular. Obviously, for the regularity level of  $z$  we have  $s_z \geq s_x$ . Only when  $s_x = s_y$ , it could happen that  $s_z > s_x$ ; when  $s_x < s_y$ , we have always that  $s_z = s_x$ .

Let us suppose that  $s_2 \geq s_x$ . Then it must be  $x_1(s_2) = -y_1(s_2)$ , from which we have that  $z = x_2(s_2) + y_2(s_2)$ . From the strictly regularity of  $x$  and  $y$ , and from Theorem 1 it follows that  $z \perp H(X; s_2)$ , that is  $z$  is regular. The regularity of the elements  $z_1(t)$  and  $z_2(t)$  follows immediately from the Consequence 1 and the above proof of the regularity of  $z$ . Hence, by Definition 3,  $z$  is strictly regular.

**Lemma 3.** If  $\{x_n\}_1^\infty$  is a convergent sequence of strictly regular elements from  $H(Y)$  and  $\lim_{n \rightarrow \infty} x_n = x$ , then  $x$  is strictly regular.

**Proof:** We shall show first a) that  $E_Y(t)x$  is regular, and then b) that  $x - E_Y(t)x$  is regular.

a)  $E_Y(t)x$  is regular if, together with all  $s$  for which  $E_Y(t)x \in H^\perp(Y; s)$ , we have that  $E_Y(t)x \in H^\perp(X; s)$ . Let us suppose that for some  $s (s < t)$  the relation  $E_Y(t)x \in H^\perp(Y; s)$ , i.e.  $E_Y(t)x \perp H(Y; s)$ , holds. Then, by reason of strictly regularity of the elements  $x_n, n = 1, 2, \dots$ , and because of Theorem 1, we have

$$0 = E_Y(s) E_Y(t)x = \lim_{n \rightarrow \infty} E_Y(s) x_n = \lim_{n \rightarrow \infty} E_X(s) x_n = E_X(s) E_Y(t)x.$$

Therefore, element  $E_Y(t)x$  is regular for all  $t$ . It remains to prove b) that  $x - E_Y(t)x = x_2(t)$  is regular. If  $x_2(t) \perp H(Y; s)$  for some  $s (s \geq t)$ , then, by the same reasons as above, we have

$$\begin{aligned} 0 &= E_Y(s) (x - E_Y(t)x) = \lim_{n \rightarrow \infty} E_Y(s) x_n - \lim_{n \rightarrow \infty} E_Y(t) x_n = \\ &= \lim_{n \rightarrow \infty} E_X(s) x_n - \lim_{n \rightarrow \infty} E_X(t) x_n = E_X(s) (x - E_Y(t)x). \end{aligned}$$

The natural question is: what could be said about the regularity level of the limit of a sequence of (strictly) regular elements? Let  $s = \overline{\lim_{n \rightarrow \infty} s_{x_n}}$ . From the sequence  $\{x_n\}_1^\infty, \lim_{n \rightarrow \infty} x_n = x$ , we can choose a subsequence  $\{x_{n_k}\}_1^\infty$  such that  $\lim_{k \rightarrow \infty} s_{x_{n_k}} = s$ . As elements  $x_{n_k}$  are (strictly) regular and  $x_{n_k} \perp H(Y; s_{n_k})$ , it follows that

$$x \perp H(Y; \lim_{k \rightarrow \infty} s_{x_{n_k}}) \text{ and } x \perp H(X; \lim_{k \rightarrow \infty} s_{x_{n_k}}),$$

i.e.  $x$  is  $s$ -regular. Therefore, we have

$$(5) \quad \lim_{n \rightarrow \infty} s_{x_n} \geq \overline{\lim_{n \rightarrow \infty} s_{x_n}}.$$

The following example shows that in (5) we can have a strict inequality.

**Example 2.** Let  $y_1$  and  $y_2$  be two arbitrarily chosen strictly regular elements, such that  $s_{y_1} < s_{y_2}$ . Let  $x_n = a_n y_1 + y_2, n = 1, 2, \dots$ , where  $\{a_n\}_1^\infty$  is a sequence of real numbers converging to zero. It is easy to see that  $\lim_{n \rightarrow \infty} x_n = s_{y_2}$

which means that  $\lim_{n \rightarrow \infty} s_{x_n} = s_{y_2}$ . However, as

$$s_{x_n} = \{\min s_{y_1}, s_{y_2}\} = s_{y_1},$$

it follows that

$$s.l. t. m. x_n = s_{y_2} > s_{y_1} = \overline{\lim}_{n \rightarrow \infty} s_{x_n} = \lim_{n \rightarrow \infty} s_{x_n}.$$

3. Let us denote by  $H^*$  the set of all strictly regular elements from  $H(Y)$ . It follows from Lemmas 2 and 3 that  $H^*$  is a Hilbert space. In order to define a process  $Y^*$ , fully submitted to  $X$  and submitted to  $Y$ , it is enough to determine the family of subspaces  $H_t^*$ ,  $0 \leq t \leq 1$ , such that  $H_t^*$  reduces the resolution of the identity  $E_X = \{E_X(s), 0 \leq s \leq 1\}$  of  $X$  and  $H_t^* \subset H(Y; t)$ ,  $0 \leq t \leq 1$ , [2, 3]. Really, if  $H_t^*$ ,  $0 \leq t \leq 1$ , is an arbitrary nondecreasing family of subspaces, a process  $Y^*$ , such that  $H(Y^*; t) = H_t^*$ ,  $0 \leq t \leq 1$ , always exists and has required properties, [1].

Let the family  $H_t^*$ ,  $0 \leq t \leq 1$ , be defined by the equality

$$H_t^* = E_Y(t) H^*, \quad 0 \leq t \leq 1.$$

Denote by  $Y^*$  the process which satisfies the equality

$$H(Y^*; t) = H_t^*, \quad 0 \leq t \leq 1.$$

Obviously,  $Y^*$  is submitted to  $Y$ , and then to  $X$ , too; let us show that it is fully submitted to  $X$ . First of all, we have that  $E_Y(t)x \in H^*$ , for any  $x \in H^*$ . Really, since  $x$  is strictly regular,  $E_Y(t)x$  is also strictly regular, and therefore  $E_Y(t)x \in H^*$ . Hence  $H_t^* \subset H^*$ . It is easy to see that any element  $x \in H^*$ , which is orthogonal to  $H_t^*$  for some  $t$ , will be also orthogonal to  $H(Y; t)$ ; really, if  $E_Y(t)x = x_1 \neq 0$ , then, by reason of  $H_t^* \subset H^*$ ,  $x_1$  belongs to  $H_t^*$ , which contradicts our assumption that  $x \in H^* \ominus H_t^*$ . Thus, the following theorem is proved.

**Theorem 2.** *Process  $Y^*$ , for which the equality*

$$H(Y^*; t) = E_Y(t) H^*, \quad 0 \leq t \leq 1,$$

*is valid, is submitted to  $Y$  and fully submitted to  $X$ .*

**Remark.** It is easy to see that the following equalities are valid:

$$H_t^* = \{y \in H^*; y \in H(Y; t)\}$$

$$H_t^{*\perp} = \{y \in H^*; y \in H^\perp(Y; t)\}.$$

The next theorem gives one interesting and not so obvious property of  $Y^*$ .

**Theorem 3.** *Let  $Y$  be an arbitrary process, submitted to  $X$ . If  $Y^*$  is a process submitted to  $Y$  and fully submitted to  $X$ , then  $Y^*$  is also fully submitted to  $Y$ .*

**Proof.** As  $Y^*$  is fully submitted to  $X$ , the space  $H(Y^*)$  could be represented as orthogonal sum of the subspaces of  $H(X)$ , cyclic with respect to  $E_X$ . If we denote the generating elements of those cyclic subspaces by  $z_\nu$ ,  $\nu \in \mathcal{N}$ , and the corresponding subspaces by  $\mathcal{M}_{z_\nu}$ ,  $\nu \in \mathcal{N}$ , then we shall have

$$H(Y^*) = \sum_{\nu} \oplus \mathcal{M}_{z_\nu} \quad \text{and} \quad H(Y^*; t) = \sum_{\nu} \oplus E_X^-(t) \mathcal{M}_{z_\nu}$$

(if  $\dim H(Y^*) = \aleph_1$ , then  $\text{card } \mathcal{A} = \aleph_1$ , and the last sum would be formal). If, for every  $\nu \in \mathcal{A}$ , we define the process  $Z_\nu$ , by

$$Z_\nu(t) = E_X(t)z_\nu, \quad 0 \leq t \leq 1,$$

then  $Z_\nu$  would be the process with orthogonal increments and the equality

$$H(Z_\nu; t) = E_X(t) \mathcal{M}_{z_\nu}, \quad 0 \leq t \leq 1,$$

will hold for any  $\nu \in \mathcal{A}$ ; it is clear that every process  $Z_\nu$  is fully submitted to  $X$ .

It is easy to see that

$$H^\perp(Z_\nu; t) = \overline{\mathcal{L}}\{Z_\nu(s) - Z_\nu(t), \quad t < s \leq 1\}, \quad 0 \leq t \leq 1.$$

But, as  $H(Y^*; t) \subset H(Y; t)$  for every  $t$ , we have  $E_X(t)z_\nu = E_Y(t)z_\nu$  for every  $t$ , and then

$$H^\perp(Z_\nu; t) = \overline{\mathcal{L}}\{(E_Y(s) - E_Y(t))z_\nu, \quad t < s \leq 1\}, \quad 0 \leq t \leq 1.$$

As  $(E_Y(s) - E_Y(t))z_\nu \in H^\perp(Y; t)$  for all  $s > t$ , then  $H^\perp(Z_\nu; t) \subset H^\perp(Y; t)$  for every  $t$ . Therefore, the process  $Z_\nu$  is, for every  $\nu$ , fully submitted to  $Y$ , which implies that  $Y^*$  is fully submitted to  $Y$ , too.

By the following example we shall show that  $Y^*$  does not always exist.

**Example 3.** Let  $\omega$  be a Wiener process defined on  $[0; 1]$ , and let

$$X(t) = \omega(t), \quad Y(t) = \omega(t^2), \quad 0 \leq t \leq 1.$$

It is clear that the equality  $H(X) = H(Y)$  holds, but, for any  $t < 1$ , we have

$$H(Y; t) = \overline{\mathcal{L}}\{\omega(u^2), \quad u \leq t\} = \overline{\mathcal{L}}\{\omega(u), \quad u \leq t^2\} = H(X; t^2) \subset H(X; t),$$

which means that  $Y$  is submitted to  $X$ . Let us show that  $Y$  is not fully submitted to  $X$ . From the assumption that  $Y$  is fully submitted to  $X$  it follows that any element  $x \in H(Y)$ , which is orthogonal to  $H(Y; t)$  for some  $t$ , will be also orthogonal to  $H(X; t)$ ; it is easy to see that such an element  $x$ , because of  $H(Y; t) = H(X; t^2)$ , can have the following form:

$$x = \int_{t^2}^1 f(u) d\omega(u),$$

where  $f$  is a function from  $L_2(dt)$ , not identically equal to zero on  $(t^2, t)$ . But, such  $x$  is not orthogonal to  $H(X; t)$ , since it is not orthogonal, for

example, to the element  $\int_{t^2}^1 f(u) d\omega(u)$ , which belongs to  $H(X; t)$ . Thus, we

proved that  $Y$  is not fully submitted to  $X$ . Let us show that  $H(Y)$  does not contain any regular element. If such an element  $z$  exists (let  $s_2$  be its regularity level), then, as follows from the previous discussion,  $z$  could be represented in the form

$$z = \int_{s_2}^1 f(u) d\omega(u),$$

where  $f(u) \neq 0$  for almost all  $u \in (s_z^2, s_z^2 + h)$ , for some  $h > 0$  (because, otherwise,  $s_z$  could not be the regularity level for  $z$ ). But we have already shown that such element is not orthogonal to  $H(X; s_z)$ , which means that it is not regular. Thus, we showed that  $H(Y)$  does not contain any regular element and the consequence of this fact is that the process  $Y^*$ , submitted to  $Y$  and fully submitted to  $X$ , does not exist.

It is clear that the necessary and sufficient condition for the process  $Y^*$  to exist is the existence of at least one regular element. Now, we shall give the necessary and sufficient conditions for the existence of  $Y^*$  in terms of the resolution of the identity of  $X$ .

**Theorem 4.** *The process  $Y^*$  exists if and only if there exists at least one element  $z \in H(Y)$ , such that*

$$(6) \quad E_X(t)z \in H(Y; t) \text{ for all } t.$$

**Proof.** Let us first suppose that  $Y^*$  exists; from this assumption it follows that in  $H(Y)$  there exists at least one strictly regular element  $z$ . For this element  $z$ , by reason of Theorem 1, the equality  $E_X(t)z = E_Y(t)z$  holds for any  $t$ , which implies that (6) holds.

Let us suppose now that there exists an element  $z$  from  $H(Y)$  such that (6) is valid. We define  $Y^*$  by

$$Y^*(t) = E_X(t)z, \quad 0 \leq t \leq 1;$$

from this definition it is clear that  $Y^*$  is fully submitted to  $X$ , and from (6) it follows that it is submitted to  $Y$ .

From Theorems 3 and 4 it follows that a necessary condition for the existence of  $Y^*$  can be expressed in terms of spectral types of  $X$  and  $Y$ .

**Theorem 5.** *Let  $\rho_X$  and  $\rho_Y$  be maximal spectral types<sup>1)</sup> of the processes  $X$  and  $Y$ , respectively. The necessary condition for the existence of  $Y^*$  is that  $\inf\{\rho_X, \rho_Y\} \neq 0$ <sup>2)</sup>.*

**Proof.** If  $Y^*$  exists, then, according to Theorem 3, it is fully submitted to both  $X$  and  $Y$ . For an arbitrary  $z$  from  $H(Y^*)$ , we have, according to Theorem 4,  $E_X(t)z = E_Y(t)z$ ,  $0 \leq t \leq 1$ , which means that  $z$  generates the same spectral type  $\rho_z$  with respect to both resolutions of the identity. Thus, we have  $\rho_z \neq 0$ ,  $\rho_z < \rho_X$  and  $\rho_z < \rho_Y$ , from where we have that  $\inf\{\rho_X, \rho_Y\} > \rho_z \neq 0$ , and our assertion is proved.

The following example will show that  $\inf\{\rho_X, \rho_Y\} \neq 0$  is not a sufficient condition for  $Y^*$  to exist.

<sup>1)</sup> We say that  $\rho_X$  is the maximal spectral type of  $X$  if any element  $z \in H(X)$  induces the spectral type  $\rho_z$  which is subordinated<sup>2)</sup> to  $\rho_X$ .

<sup>2)</sup> We say that the spectral type  $\rho$  is subordinated to spectral type  $\mu$ , and we write  $\rho < \mu$ , if any measure of the spectral type  $\rho$  is absolutely continuous with respect to any measure of the spectral type  $\mu$ .

<sup>3)</sup> By  $\inf\{\rho, \mu\}$  we denote the greatest spectral type which is subordinated to  $\rho$  and  $\mu$ .

Example 4. Let the processes  $X$  and  $Y$  be defined as in Example 3. Both processes are Wiener, so that they have a unit multiplicities and the spectral types  $\rho_X$  and  $\rho_Y$  are equal to the ordinary Lebesgue measure. Therefore,  $\inf \{\rho_X, \rho_Y\} \neq 0$ , although  $Y^*$  does not exist, as it is shown in Example 3.

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