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CONDITIONS FOR THE INTEGRABILITY OF A SECOND ORDER NONLINEAR DIFEERENTIAL EQUATION

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0. The object of this paper is investigation of integrability of differential equation of the from:

$$y''+A(y,\,x)\,y'+B(y,\,x)=0, \tag{0.1}$$

where A, B are given functions.

A number of authors (see [1] — [18]) considered some special cases of the above equation. Only in Kamke's collection [1], are noted 72 equations of the form (0.1). Also, Painlevé [2] considered some equations of the type (0.1).

Important special cases of (0.1), are the well known, generalized Emden's equations:

$$y''+v(x)\,y'+w(x)\,y^n=0, \qquad (n\in R,\ n\neq 0,\ 1) \tag{0.2}$$

$$y''+v(x)\,y'+w(x)\,e^y=0, \tag{0.3}$$

(v, w are given functions). Those equations are considered in papers [6] — [12]. For the above equations the following results are known (see, for example [7])

1° if

$$w(x)=\alpha\exp\left(-2\int v(x)\,dx\right) \qquad (\alpha=\text{const.}) \tag{0.4}$$

then equation (0.2) and (0.3) are integrable by quadratures;

2° if

$$w(x)=\alpha\exp\left(-2\int v(x)\,dx\right)\left(\int\exp\left(-\int v(x)\,dx\right)dx\right)^{-n-3}, \tag{0.5}$$

$$w(x)=\alpha\exp\left(-2\int v(x)\,dx\right)\exp\left(\int\exp\left(-\int v(x)\,dx\right)dx\right), \tag{0.6}$$

(α = const.) then equations (0.2) (0.3), respectively, are integrable by quadratures.

In papers [7], [13] — [16] authors solved some nonlinear differential equations which can be reduced to an autonomous form. They, in fact, considerd differential equations $N(y(x), x)=0$, which, after transformation $y=q(x)z(t)$, $dt=p(x)dx$, have the autonomous form $M(z(t))=0$.

In section 1 we ask the following question:

When does the equation (0.1) after the transformation $y=F(Y(X), X)$, $dX=a(X)dx$, reduce to the autonomous form.

Theorem 1 gives the comlete answer to the above question. The general form of equations which have this property is given by theorem 2.

Our results are compared with some known in section 2.

The above results are applied to equtions (0.2) and (0.3) in sections 3 and 4, respetively. Conditions (3.10) and (4.8), which ensure the integrability in quadratures of (0.2) and (0.3), respectively, are more general then those already known.

Equation

(0.7) $$y''+v(x)y+w(x)y^n=0, \qquad (n\in R)$$

is treated in section 5. This equation is considered in papers [17] and [18].

1. We consider the following problem:

Find the conditions under which the equation (0.1) reduces to an equation of the form

(1.1) $$\frac{d^2 Y}{dX^2}=\varphi\left(\frac{dY}{dX}, Y\right),$$

by the transformation

(1.2) $$y(x)=F(Y(X), X) \quad dx=\frac{dX}{a(X)},$$

($Y(X)$, X are new unknown function and new variable, respectively; F is twice continuously differentiable function and a is continuously differentiable function).

From (1.2) we find y' and y'' and after the substitution in (0.1), we obtain the equation

$$\frac{d^2 Y}{dX^2}+\frac{F_{YY}}{F_Y}\left(\frac{dY}{dX}\right)^2+\left(2\frac{F_{YX}}{F_y}+\frac{a'}{a}+\frac{1}{a}A\left(F, \int\frac{dX}{a}\right)\right)\frac{dY}{dX}$$
$$+\frac{F_{XX}}{F_Y}+\frac{a'}{a}\frac{F_X}{F_Y}+\frac{1}{a}A\left(F, \int\frac{dX}{a}\right)\frac{F_X}{F_Y}+\frac{1}{a^2}B\left(F, \int\frac{dX}{a}\right)=0.$$

The above equation has the from (1.1) if and only if there exist functions P, Q, R such that:

(1.3) $$\frac{F_{YY}}{F_Y}=P(Y),\ 2\frac{F_{YX}}{F_Y}+\frac{a'}{a}+\frac{1}{a}A\left(F, \int\frac{dX}{a}\right)=Q(Y),$$
$$\frac{F_{XX}}{F_Y}+\frac{a'}{a}\frac{F_X}{F_Y}+\frac{1}{a}A\left(F, \int\frac{dX}{a}\right)\frac{F_X}{F_Y}+\frac{1}{a^2}B\left(F, \int\frac{dX}{a}\right)=R(Y),$$

and the same equation becomes:

$$(1.4)\qquad \frac{d^2 Y}{dX^2}+P(Y)\left(\frac{dY}{dX}\right)^2+Q(Y)\frac{dY}{dX}+R(Y)=0.$$

Furthermore, from the first equality from (1.3) it follows that F has the form $F(Y, X)=b(X)\int \exp\left(\int P(Y)dY\right)dY+c(X)$ (b, c are twice continuosly differentiable functions of X).

If we introduce the substitution $z(t)=\int \exp\left(\int P(Y)dY\right)dY$, $t=X$, where z and t are new unknown function and new variable, respectively, then equation (1.4) becomes

$$(1.5)\qquad \frac{d^2 z}{dt^2}+f(z)\frac{dz}{dt}+g(z)=0$$

where f and g are given by

$$(1.6)\qquad f\left(\int \exp\left(\int P(Y)\,dY\right)dY\right)=Q(Y),$$

$$g\left(\int \exp\left(\int P(Y)\,dY\right)dY\right)=R(Y)\exp\left(-\int P(Y)\,dY\right).$$

Then we have

$$(1.7)\qquad y(x)=b(t)\,z(t)+c(t),\quad dt=a(t)\,dx.$$

In accordance with the above we can formulate the following result:

Theorem 1. *The equation* (0.1), *after the transformation* (1.2) *reduce to the autonomous form* (1.1), *if and only if after the linear transformation* (1.7) *it has the autonomous form* (1.5).

We see that our problem is reduced to detemining equations which transform to autonomous form under the linear transformation (1.7).

Furthermore, we shall determine the form of functions A and B.

Putting $a(t)=p(x)$, $b(t)=q(x)$, $c(t)=r(x)$, we find that A, B have the form

$$(1.8)\qquad A(y,x)=pf\left(\frac{y-r}{q}\right)-2\frac{q'}{q}-\frac{p'}{p},$$

$$B(y,x)=p^2 qg\left(\frac{y-r}{q}\right)-\left(\frac{q''}{q}+\frac{q'}{q}A(y,x)\right)(y-r)-r''-r''A(y,x).$$

Now, we can formulate the following theorem:

Theorem 2. *Equation* (0.1) *can be reduced to the autonomous from* (1.1), *under she transformation* (1.2), *if and only if functions A and B have the form* (1.8). *In this case after the substitution*

$$(1.9)\qquad y(x)=q(x)\,z(t)+r(x),\ dt=(p(x)\,dx,$$

(0.1) *reduces to equation* (1.5).

Furthermore, putting $\frac{dz}{dt}=u(z)$, equation (1.5) becomes:

$$u(z)\frac{du}{dz}+f(z)\,u(z)+g(z)=0. \tag{1.10}$$

This means that eqution (0.1), where A, B are given by (1.8), has the general solution:

$$\int\frac{d\left(\frac{y-r}{q}\right)}{U\left(\frac{y-r}{q},\,C_1\right)}=\int p(x)\,dx+C_2, \tag{1.11}$$

where C_1, C_2 are arbitrary constants and $u=U(z, C_1)$ is the general solution of the first order equation (1.10).

2. In this section we shall compare our results to some known results. Also, we point out some particular cases.

1° Let $r(x)\equiv 0$. Then (1.8) becomes:

$$A(y, x)=pf\left(\frac{y}{q}\right)-2\frac{q'}{q}-\frac{p'}{p},\quad B(y, x)=p^2qg\left(\frac{y}{q}\right)-\left(\frac{q''}{q}+\frac{q'}{q}A(y, x)\right)y. \tag{2.1}$$

2° If $q(x)\equiv 1$, then (1.8) has the form:

$$A(y, x)=pf(y-r)-\frac{p'}{p}\quad B(y, x)=p^2g(y-r)-r''-r'A(y, x). \tag{2.2}$$

3° Differential equation (0.1) is noted in [1] (eq. 6.44), and the following result is given:

If functions A and B satisfy the condition

$$B_y(y, x)-A_x(y, x)=V(x)A(y, x)-V(x)^2-V'(x), \tag{2.3}$$

where V is some function of x, then equation (0.1) can be reduced to a first order equation.

It is easy to see that functions A and B given by (1.8) do not satisfy the condition (2.3), i.e. our result and the above cannot be compared to each other.

4° In [3] J. D. Kečkić considered the following equation:

$$y''+v(x)\,y'+w(x)\,h(y)=0 \tag{2.4}$$

(v, w, h are given functions).

Putting $f \equiv 0$, $q \equiv 1$, $p = \exp\left(-\int v(x)\,dx\right)$, $h = g$, $A(y, x) = v(x)$, $B(y, x) = w(x)\,h(y)$, $w(x) = C\exp\left(-2\int v(x)\,dx\right)$, we see that, in this case, conditions (2.1) are sattsfied and equation (2.4) is integrable.

The same result was obtained in [3] by applying a variant of the variation of parameters method. This method J. D. Kečkić applied in [4] to some more general second order equations.

5° D. S. Mitrinović and J. D. Kečkić [5] consider differential equations of the form

$$(2.5) \qquad y'' + A(x)\,y' + B(y, x) = 0.$$

If we take $f \equiv 0$, $q = \exp\left(-\int \varphi\,dx\right)$, $p = \exp\left(\int (\varphi - \psi)\,dx\right)$, $g = h$, $A(y, x) = A(x)$, $B(y, x) = \exp\left(\int (\varphi - 2\psi)\,dx\right) h\left(y \exp\left(\int \varphi\right) dx\right)$, where functions φ and ψ satisfy

$$\varphi(x) + \psi(x) = A(x), \quad \varphi'(x) + \varphi(x)\,\psi(x) = 0,$$

then conditions (2.1) are fulfilled and we have that (2.5) is integrable by quadratures.

The same result was obtained in paper [5].

6. Equation (0.1) containes in special cases, 72 equations from Kamke's collection [1]. Those are equations (6.1) — (6.44), (6.73) — (6.78), (6.82) — (6.84), (6.91), (6.92), (6.94) — (6.97), (6.100) — (6.102), (6.104) — (6.106), (6.108), (6.188), (6.205), (6.209), (6.211), (6.219) — (6.220). Many of these equations can be solved by applying the above method.

3. Now we consider the generalized Emden's equation (0.2).

In papers [6] — [12] were given a number of conditions which ensure the integrability of (0.2). In [7] L. M. Berkovič showed that all these conditions are equivalent to (0.4) or (0.5). In this section we give a condition which containes (0.4) and (0.5) as particular cases.

From theorem 2 it follows that, after the transformation (1.9), with $r(x) \equiv 0$, equation (0.2) has the form (1.5) if and only if:

$$(3.1) \qquad v(x) = pf\left(\frac{y}{q}\right) - 2\frac{q'}{q} - \frac{p'}{p}, \quad w(x)\,y^n = p^2 gq\left(\frac{y}{q}\right) - \left(\frac{q''}{q} + \frac{q'}{q}v(x)\right)y.$$

Then we have $f(z) = \text{const.}$, and taking $f(z) \equiv 0$, we find

$$pq^2 = K\exp\left(-\int v(x)\,dx\right) \quad (K = \text{const.} \neq 0).$$

Furthermore, second condition from (3.1) implies

$$(3.2) \qquad g(z) = \alpha z^n + \beta z \quad (\alpha, \beta = \text{const.}, \ \alpha \neq 0),$$

and we hawe $w(x)\,y^n = \alpha p^2 q^{1-n} y^n + \left(\beta p^2 - \frac{q''}{q} - \frac{q'}{q}v(x)\right)y$, i.e.

$$w(x) = \alpha p^2 q^{1-n} = \alpha K^2 \exp\left(-2\int v(x)\,dx\right) q^{-3-n}, \quad \beta p^2 - \frac{q''}{q} - \frac{q'}{q}v(x) = 0.$$

Accordingly, we obtain that q satisfy the following equation

$$q''+v(x)\,q'=\alpha K^2\exp\left(-2\int v(x)\,dx\right)q^{-3}. \tag{3.3}$$

Taking $K=1$ we have that the general solution of (3.3) can be expressed as (see [19]):

$$q(x)=\left(C_1+C_2\int e^{-\int v dx}dx+C_3\left(\int e^{-\int v dx}dx\right)^2\right)^{1/2}, \tag{3.4}$$

where C_1, C_2, C_3 are arbitrary constants such that

$$C_1C_3=\beta+C_2^2/4. \tag{3.5}$$

Then we have

$$w(x)=\alpha e^{-2\int v dx}\left(C_1+C_2\int e^{-\int v dx}dx+C_3\left(\int e^{-\int v dx}dx\right)^2\right)^{-(n+3)/2}. \tag{3.6}$$

Then equation (1.7) has the form:

$$\frac{d^2z}{dt^2}+\alpha z^n+\beta z=0, \tag{3.7}$$

and the general solution of the above is given by

$$\begin{cases}\int\left(C-\beta z^2-\frac{2\alpha}{n+1}z^{n+1}\right)^{-1/2}dz=t+D & (n\neq-1)\\ \int(C-\beta z^2-2\alpha\log z)^{-1/2}dz=t+D & (n=-1)\end{cases} \tag{3.8}$$

(C, D are arbitrary constants).

Hence we can formulate the result:

Theorem 3. *Equation* (0.2) *is integrable if function* w *has the form* (3.10). *In this case the general solution if* (0.2) *is* $y(x)=q(x)\,z\left(\int p(x)\,dx\right)$, *where* z *is given by* (3.8), $p=\exp\left(-\int v(x)\,dx\right)q^{-2}$, q *is given by* (3.4) *with* (3.5).

Remarks. 1° Putting in (3.5) $C_1=C_2=0$, $C_3=1$ we find $\beta=0$ and (3.6) becomes (0.5).

2° If we take $C_2=C_3=0$, $C_1=1$ we obtain $\beta=0$ and (3.6) reduces to (0.4).

3° Let $C_1=C_3=0$, $C_2=2$. Then $\beta=-1$ and (3.6) becomes:

$$w(x)=\alpha\exp\left(-2\int v(x)\,dx\right)\left(2\int\exp\left(-\int v(x)\,dx\right)dx\right)^{-(n+3)/2}$$

4° In the case $C_1=\gamma^2$, $C_2=2\gamma$, $C_3=1$, ($\gamma=$ const.) we have $\beta=0$ and

$$w(x)=\alpha\exp\left(-2\int v(x)\,dx\right)\left(\int\exp\left(-\int v(x)\,dx\right)dx+\gamma\right)^{-(n+3)}.$$

5° Now we consider the well known Emden-Fowler's equation (see, for example [7])

$$y''+\frac{a}{x}y'+bx^{m-1}y^n=0 \quad (m,\ n\in R;\ a,\ b=\text{const.};\ a\neq 1).$$

For $v(x)=a/x$ and $w(x)=bx^{m-1}$ from (3.6) it follows that the following condition must be satisfied:

$$bx^{m-1}=\alpha\, x^{-2a}\left(C_1+\frac{C_2}{1-a}\,x^{1-a}+\frac{C_3}{(1-a)^2}\,x^{2-2a}\right)^{-(n+3)/2}.$$

This is possible only in the following thee casses:

(i) $C_2=C_3=0;\ 2a=1-m;\ C_1=(\alpha/b)^{2/(n+3)}$;

(ii) $C_1=C_3=0;\ (n-1)(a-1)=2m+2;\ C_2=(1+a)(\alpha/b)^{2/(n+3)}$;

(iii) $C_1=C_2=0;\ (n+1)(a-1)=m+1;\ C_3=(1-a)^2(\alpha/b)^{2/(n+3)}$.

Conditions (i), (ii), (iii) are noted by L. M. Berkovič [7].

6° Applying a similar procedure to the equation

$$y''+v(x)\,y'+\sum_{i=1}^{k}w_i(x)y^{n_i}=0 \tag{3.9}$$

($n_i\in R$, $i=1,\dots k$; $v, w_1,\dots, w_k$ are given functions), we obtain that, if

$$w_i(x)=\alpha_i e^{-2\int v dx}\left(C_1+C_2\int e^{-\int v dx}dx+C_3\left(\int e^{-\int v dx}dx\right)^2\right)^{-(n_i+3)/2}.$$

($\alpha_i=$ const.; C_1, C_2, C_3 are constants such that (3.5) holds), then equation (3.9) is integrable by quadratures.

4. Let us consider the generalized Emden's equation (3.3).

This equation is treated by a number of authors (see [7] — [12]). Various conditions, which ensure the integrability of this equaion, in [7] — [12], are equivalent to (0.4) and (0.6) (see [7]). In this section we give a more general condition than (0.4) and (0.6).

From the theorem 2 it follows that the transformation (1.9) with $q(x)=1$, reduces equation (0.3) to an equation of the from (1.5) if and only if

$$v(x)=pf(y-r)-\frac{p'}{p},\quad w(x)\,e^y=p^2 g(y-r)-r''-r'v(x). \tag{4.1}$$

Then we have $f(z)=$ const., and if we take $f(z)=0$, we obtain $p(x)=$ $=K\exp\left(-\int v(x)\,dx\right)$ ($K=$ const. $\neq 0$). Furthermore, from the second equality in (4.1) we find

$$g(z)=\alpha\, e^z+\beta \quad (\alpha,\ \beta=\text{const.},\ \alpha\neq 0). \tag{4.2}$$

This implies that $w(x)e^y = \alpha p^2 e^{-r} e^y - r'' - r' v(x) + \beta p^2$, i.e,

$$w(x) = \alpha p^2 e^{-r} = \alpha K^2 e^{-r-2\int v dx}, \quad r'' + v(x) r' = \beta K^2 e^{-2\int v dx}. \tag{4.3}$$

Taking $K=1$, we obtain that the general solution of equation (4.3) is given by:

$$r(x) = C_1 + C_2 \int e^{-\int v dx} dx + \frac{\beta}{2}\left(\int e^{-\int v dx} dx\right)^2 \tag{4.4}$$

and

$$w(x) = \alpha \exp\left(-2\int v(x)\,dx - C_1 - C_2 \int e^{-\int v dx} dx - \frac{\beta}{2}\left(\int e^{-\int v dx} dx\right)^2\right). \tag{4.5}$$

Then equation (1.5) becomes:

$$\frac{d^2 z}{dt^2} + \alpha e^z + \beta = 0, \tag{4.6}$$

which has the general solution:

$$\int (2\alpha e^z + 2\beta z + C)^{-1/2} dz = t + D \tag{4.7}$$

(C, D are arbitrary constants).

Now, we can formulate the following theorem

T h e o r e m 4. *Equation* (0.3) *is integrable by quadratures if the function* w *has the form* (4.5). *The general solution of this equation is then given by* $y(x) = z\left(\int p(x)\,dx\right) + r(x)$, *where* z *is given by* (4.7), $p = \exp\left(-\int v(x)\,dx\right)$, r *is defined by* (4.4).

Remarks. 1° Let $\beta=0$, $C_1=0$, $C_2=-1$. Then (4.5) reduces to (0.6).

2° Putting $\beta = C_1 = C_2 = 0$, (4.5) becomes (0.4).

3° In the case $C_1 = C_2 = 0$, $\beta = -2$, from (4.5) we have

$$w(x) = \alpha \exp\left(-2\int v(x)\,dx + \left(\int \exp\left(-\int v(x)\,dx\right) dx\right)^2\right).$$

4° For the equation:

$$y'' + v(x) y' + \sum_{i=1}^{k} w_i(x) e^{a_i y} = 0 \tag{4.8}$$

($a_1, \ldots, a_k \in R$; $v, w_1, \ldots, w_k$ are given functions), we can obtain the following conditions which ensure its integrability:

$$w_i(x) = \alpha_i \exp\left(-2\int v(x)\,dx - a_i r(x)\right) \quad (i = 1, \ldots, k),$$

where r is given by (4.4).

5° Equation

$$y'' + v(x) y' + w_1(x) ch\, y + w_2(x) sh\, y = 0,$$

is a special case of (4.8). If $w_1(x) = \exp(-2 v(x)\, dx)(A_1 e^{-r(x)} + A_2 e^{r(x)})$ $w_2(x) = \exp\left(-2 \int v(x)\, dx\right)(A_1 e^{r(x)} + A_2 e^{r(x)})$, ($A_1, A_2 =$ const., r is given by (4.9)) then the above equation is integrable.

6° Similarly we can consider the following equation:

$$y'' + v(x) y' + w_1(x) \cos y + w_2(x) \sin y = 0,$$

which is integrable, if $w_1(x) = \exp\left(-2 \int v(x)\right)(A_1 \cos r(x) - A_2 \sin r(x))$, $w_2(x) = \exp\left(-2 \int v(x)\right)(A_1 \sin r(x) + A_2 \cos r(x))$, where A_1, A_2 are constants and r is given by (4.4).

5. Let us consider the equation (0.7). After the substitution $Y(x) = y(x) \exp\left(-\frac{1}{2} \int \Phi(x)\, dx\right)$, where Φ is a solution of Riccati's differential equation

$$\Phi' + \frac{1}{2} \Phi^2 + 2 v(x) = 0, \tag{5.1}$$

(0.7) reduces to generalized Emden's equation:

$$Y'' \Phi(x) Y' + \psi(x) Y^n = 0. \tag{5.2}$$

Function Ψ is given by

$$\Psi(x) = w(x) \exp\left(\frac{n-1}{2} \int \Phi(x)\, dx\right). \tag{5.3}$$

In section 3 a sufficient condition for integrability of generalized Emden's equation is given. Using this condition and (5.3) we find that equation (0.7) is integrable if

$$w(x) = \alpha \left(e^{-\int \Phi\, dx} \left(C_1 + C_2 \int e^{-\int \Phi\, dx} dx + C_3 \left(\int e^{-\int \Phi\, dx} dx\right)^2\right)^{-(n+3)/2},\right.$$

where α, C_1, C_2, C_3 are constants such that (3.5) holds.

The general solution of (0.7) is given by $y(x) = Y(x) \exp\left(\frac{1}{2} \int \Phi(x)\, dx\right)$, where Y is the general solution of (5.1) (see section 3).

Remark. 1° In [17] L. M. Berkovič and N. H. Rozov proved the following result for the equation (0.7):

Equation (0.7) is integrable in quadratures if function w has the form $w(x) = \alpha s(x)^{-(n+3)}$, where s is a solution of Pinney's equation (see [18], [19])

$$s'' + v(x) s = \beta s^{-3}.$$

It can be shown that the above result and our are equivalent.

2° If we take $C_1 = C_3 = 0$, $C_2 = 1$ we obtin the result of J. L. Reid from [18].

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