PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 27 (41) 1980, pp. 107—112

ON FIXED POINT THEOREMS IN 2-METRIC SPACE

by

M. S. Khan

1. Introduction:

The notion of a 2-metric space was introduced by S. Gähler [2]. Recently the theory of 2-metric spaces has been extensively studied and developed by Gähler [2] – [4], White [9], Iséki ([5], [6]) and Iséki-Sharma-Sharma [7].

In this note we intend to prove some fixed point theorems in 2-metric spaces and also discuss the convergence of sequences of mappings and fixed points in such spaces. Our main theorem generalizes a theorem of Iséki [6].

2. Preliminaries:

Following Gähler [2] and White [9] we have following definitions:

Definition 2.1 — A *2-metric space* is a space X in which, for each triple of points a, b, c there exists a real valued function $\rho(a, b, c)$ such that

(i) to each pair of points a, b with $a \neq b$ from X, there exists a point $c \in X$ such that $\rho(a, b, c) \neq 0$,

(ii) $\rho(a, b, c) = 0$ when at least two of the points are equal,

(iii) $\rho(a, b, c) = \rho(b, c, a) = \rho(a, c, b)$

(iv) $\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$.

It is easily seen that ρ is non-negative.

Definition 2.2 — A 2-metric space is called *bounded*, if there exists a constant K such that $\rho(a, b, c) \leq K$ for $a, b, c \in K$.

Definition 2.3 — A sequence $\{x_n\}$ in X is *convergent* and $x \in X$ is the limit of this sequence if $\lim_{n \to \infty} \rho(x_n, x, a) = 0$ for all $a \in X$.

Definition 2.4 — A sequence $\{x_n\}$ in a 2-metric space X is called a *Cauchy sequence* if $\lim_{m, n \to \infty} \rho(x_m, x_n, a) = 0$ for all $a \in X$.

If every Cauchy sequence is convergent X is called complete 2-metric space.

Definition 2.5 — Let T be a mapping of a 2-metric space X into itself. If for all $a \in X$,

$$\rho(T^n x, u, a) \to 0 \quad (n \to \infty)$$

implies

$$\rho(TT^n x, Tu, a) \to 0 \quad (n \to \infty)$$

then T is called *orbitally continuous* ([5]).

Definition 2.6 — A 2-metric ρ which is continuous in all of its three arguments is called continuous.

3. Main results:

Iséki [6] proved the following fixed point theorem for 2-metric space which was obtained earlier by Ćirić [1] for 1-metric space.

Theorem A — *Let T be an orbitally continuous mapping of a bounded complete 2-metric space X into itself. If T satisfies the condition:*

$$\min\{\rho(Tx, Ty, a), \ \rho(x, Tx, a), \ \rho(y, Ty, a)\}$$
$$-\min\{\rho(x, Ty, a), \ \rho(y, Tx, a)\} \leq q\,\rho(x, y, a)$$

for all $x, y, a \in X$, and for some q with $0<q<1$, then for each $x \in X$, the sequence $\{T_x^n\}$ $(n=1, 2, 3, \ldots)$ converges to a fixed point of T.

In this paper we prove the following generalization of Theorem A.

Theorem 3.1 — *Let T be an orbitally continuous mapping of a complete bounded 2-metric space X into itself If T satisfies the condition: there exist real numbers $\alpha_1, \alpha_2, \alpha_3, \beta$ for every $x, y, a \in X$ such that*

$$\alpha_1+\alpha_2+\alpha_3>\beta, \ \beta-\alpha_2 \geq 0, \ \beta-\alpha_3 \geq 0$$

and

(*) $$\alpha_1\,\rho(Tx, Ty, a)+\alpha_2\,\rho(x, Tx, a)+\alpha_3\,\rho(y, Ty, a)$$
$$-\min\{\rho(x, Ty, a), \ \rho(y, Tx, a)\} \leq \beta\rho(x, y, a)$$

for all $x, y, a \in X$, then for each $x \in X$, the sequence $\{T^n x\}$ converges to fixed point of T.

Proof — Let $x_0 \in X$ and define $x_{n+1}=T(x_n)$, $n=0, 1, 2, 3, \ldots$. Since $x_n=x_{n+1}$ for all n immediately implies that $\{x_n\}$ is a Cauchy sequence and the limit of $\{x_n\}$ is a fixed point of T.

Suppose that $x_n \neq x_{n+1}$ for every $n=0, 1, 2, \ldots$. By (*) for $x=x_{n-1}$ and $y=x_n$ we have

$$\alpha_1\,\rho(x_n, x_{n+1}, a)+\alpha_2\,\rho(x_{n-1}, x_n, a)+\alpha_3\,\rho(x_n, x_{n+1}, a)$$
$$-\min\{\rho(x_{n-1}, x_{n+1}, a), \ \rho(x_n, x_n, a)\} \leq \beta\rho(x_{n-1}, x_n, a).$$

Then we get

$$\rho(x_n, x_{n+1}, a) \leq \left(\frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right) \rho(x_{n-1}, x_n, a).$$

Proceeding in this manner we obtain

$$\rho(x_n, x_{n+1}, a) \leq q^n \rho(x_0, x_1, a)$$

where

$$q = \left(\frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right) < 1.$$

Then as in the proof of Theorem A ([6]) we conclude that $\{x_n\}$ is a Cauchy sequence which must converge to some point $u \in X$. Hence from [5] we observe that u is a fixed point of T.

Remarks:

(a) When the mapping T is *not* orbitally continuous we proceed as follows:

$$\alpha_1 \rho(Tu, Tx_n, a) + \alpha_2 \rho(u, Tu, a) + \alpha_3 \rho(x_n, Tx_n, a)$$
$$- \min\{\rho(u, Tx_n, a),\ \rho(x_n, Tu, a)\} \leq \beta\rho(u, x_n, a)$$

where taking $n \to \infty$,

$$\alpha_1 \rho(Tu, u, a) + \alpha_2 \rho(u, Tu, a) \leq 0$$

or

$$(\alpha_1 + \alpha_2) \rho(Tu, u, a) = 0.$$

Here

$$\alpha_1 + \alpha_2 > 0 \quad \text{as} \quad \beta - \alpha_3 \geq 0.$$

So

$$\rho(Tu, u, a) = 0 \quad \text{for all} \quad a \in X.$$

If $Tu \neq u$, then by (i) of definition 1, $(Tu, u, a) \neq 0$ for some $a \in X$ which is inadmissible. Therefore $Tu = u$.

(b) Theorem 3.1 for 1-metric space was proved by Tasković [8].

(c) It has been demonstrated in [8] that 1-metric space version of Theorem 3.1 is actually a generalization of 1-metric space version of Theorem A. Thus Theorem 3.1 generalizes Theorem A in 2-metric space.

(d) As in [8] the proof of Theorem 3.1 is made under the assumption that $\beta - \alpha_2 \geq 0$ implying there by $\alpha_1 + \alpha_3 > 0$.

Theorem 3.2 — *Let T_1 and T_2 be mappings of a complete bounded 2-metric space X into itself satisfying:*

(**)
$$\alpha_1 \rho(T_1 x, T_2 y, a) + \alpha_2 \rho(x, T_1 x, a) + \alpha_3 \rho(y, T_2 y, a)$$
$$- \min\{\rho(x, T_2 y, a),\ \rho(y, T_1 x, a)\} \leq \beta\rho(x, y, a)$$

for all $x, y, a \in X$, where α_1, α_2, α_3, β are as in Theorem 3.1. Then T_1 and T_1 have a common fixed point.

Proof. — Let $x_0 \in X$ and define $\{x_n\}$ by $x_{2n+1} = T_1(x_{2n})\, x_{2n+2} = T_2(x_{2n+1})$. Then by routine calculation we find that

$$\rho(x_n, x_{n+1}, a) \leq q^n \rho(x_0, x_1, a)$$

with

$$q = \left(\frac{\beta - \alpha_2}{\alpha_1 + \alpha_3}\right) < 1.$$

Now it follows as in the proof of Theorem A [6], that $\{x_n\}$ is a Cauchy sequence which has some limit say $u \in X$. Then

$$\rho(T_1 u, u, a) \leq \rho(T_1 u, u, x_{2n+2}) + \rho(T_1 u, x_{2n+2}, a) + \rho(x_{2n+2}, u, a).$$

From (**),

$$\alpha_1 \rho(T_1 a, x_{2n+2}, a) + \alpha_2 \rho(u, T_1 u, a) + \alpha_3 \rho(x_{2n+1}, x_{2n+2}, a)$$
$$- \min\{\rho(u, a_{2n+2}, a),\ \rho(x_{2n+1}, T_1 u, a)\} \leq \beta \rho(u, x_{n+2}, a)$$

which yields by letting $n \to \infty$,

$$(\alpha_1 + \alpha_2)\rho(T_1 u, u, a) \leq 0.$$

We find $\rho(T_1 u, u, a) = 0$ for all $a \in X$. Hence $T_1 u = u$.
Similarly $T_2 u = u$. Thus u is a common fixed point of T_1 and T_2.

Theorem 3.3 — *Let X be a complete bounded 2-metric space, $\{T_n\}$, $n = 1, 2, 3, \ldots$ a sequence of mappings of X into itself such that for all $x, y, a \in X$:*

$$\alpha_1 \rho(T_1 x, T_j y, a) + \alpha_2 \rho(x, T_i x, a) + \alpha_3 \rho(y, T_j y, a)$$
$$- \min\{\rho(x, T_j y, a),\ \rho(y, T_i x, a)\} \leq \beta \rho(x, y, a)$$

where α_1, α_2, α_3 and β are as in Theorem 3.1. Then the sequence $\{T_n\}$ has a common fixed point.

Proof — For any arbitrary $x_0 \in X$ define $\{x_n\}$ by $x_n = T_n(x_{n-1})$, $n = 1, 2, 3, \ldots$. Then by simple calculation $\{x_n\}$ is found to be a Cauchy sequence with limit $u \in X$.

Now

$$\alpha_1 \rho(T_n u, T_{m+1} x_m, a) + \alpha_2 \rho(u, T_n u, a)$$
$$+ \alpha_3 \rho(x_m, x_{m+1}, a) - \min\{\rho(u, x_{m+1}, a),\ \rho(x_m, T_n u, a)\}$$
$$\leq \beta \rho(u, x_m, a).$$

Letting $m \to \infty$, we have

$$\alpha_1 \rho(T_n u, u, a) + \alpha_2 \rho(u, T_n u, a) \leq 0$$

or

$$(\alpha_1 + \alpha_2)\rho(T_n u, u, a) = 0.$$

Theorem 3.4 — *Let X be a complete bounded 2-metric space with ρ continuous, $\{T_n\}$ a sequence of mappings of X into itself satisfying (*) of Theorem 3.1 for each n and same α_i, β such that $\{T_n\}$ converges pointwise to a orbitally continuous function T. Then f has a fixed point Z and $Z_n \to Z$ where Z_n are fixed points of T_n.*

Proof — For each $x, y \in X$ we have from (*),

$$\alpha_1 \rho (T_n x, T_n y, a) + \alpha_2 \rho (x, T_n x, a) + \alpha_3 \rho (y, T_n y, a)$$
$$- \min \{\rho (x, T_n y, a), \ \rho (y, T_n x, a)\} \leq \beta\rho (x, y, a).$$

Taking the limit of both sides as $n \to \infty$, and using the continuity of ρ, we find that T satisfies (*). Hence T has a fixed point, say z.

Now by inequality (iv) of Definition 2.1,

$$\rho (z, z_n, a) = \rho (Tz, T_n z_n, a)$$
$$\leq (Tz, T_n z_n, T_n z) + \rho (Tz, T_n z, a)$$
$$+ \rho (T_n z, T_n z_n, a).$$

Again from (*)

$$\alpha_1 \rho (T_n z, T_n z_n, a) + \alpha_2 \rho (z, T_n z, a) + \alpha_3 \rho (z_n, T_n z_n, a)$$
$$- \min \{\rho (z, T_n z_n, a), \ \rho (z_n, T_n z, a)\} \leq \beta\rho (z, z_n, a)$$

or

$$\alpha_1 \rho (T_n z, T_n z_n, a) + \alpha_2 \rho (z, T_n z, a) - \min \{\rho (z, z_n, a), \ \rho (z_n, T_n z, a)\}$$
$$\leq \beta\rho (z, z_n, a).$$

Two cases arise

(i) When $\rho (z, z_n, a)$ is the minima, we get

$$\rho (T_n z, T_n z_n, a) \leq \left(\frac{\beta + 1}{\alpha_1}\right) \rho (z, z_n, a) - \frac{\alpha_2}{\alpha_1} \rho (z, T_n z, a)$$

(ii) When $\rho (z_n, T_n z, a) = \rho (T_n z_n, T_n z, a)$ is the minima one obtains

$$\rho (T_n z, T_n z_n, a) \leq \left(\frac{\beta}{\alpha_1 - 1}\right) \rho (z, z_n, a) - \rho \left(\frac{\alpha_2}{\alpha_1 - 1}\right) \rho (z, T_n z, a).$$

Then

$$\rho (Tz, T_n z_n, T_n z) = \rho (T_n z_n, T_n z, Tz)$$
$$\leq \left(\frac{\beta}{\alpha_1 - 1}\right) \rho (z, z_n, Tz) - \left(\frac{\alpha_2}{\alpha_1 - 1}\right) \rho (z, T_n z, Tz)$$
$$= 0.$$

Thus we get

$$\rho (z, z_n, a) \leq \rho (z, T_n z, a) + \left(\frac{\beta + 1}{\alpha_1}\right) \rho (z, z_n, a) - \frac{\alpha_2}{\alpha_1} \rho (z, T_n z, a)$$

or

$$\rho (z, z_n, a) \leq \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 - \beta - 1}\right) \rho (z, T_n z, a).$$

Here right hand side tends to 0 as $n \to \infty$ proving that $z_n \to z$.

Theorem 3.5 — *Let X be a bounded complete 2-metric space $\{T_n\}$ a sequence of mappings of X into X with fixed points z_n, and T is a mapping of X into X satisfying (*) of Theorem* 3.1 *with fixed point z, such that $T_n \to T$ uniformly on $\{z_n : n=1, 2, \ldots\}$. Then $z_n \to z$.*

Proof — Firstly we have,

$$\rho(z_n, z, a) = \rho(T_n z_n, Tz, a)$$
$$\leq \rho(T_n z_n, Tz, Tz_n) + \rho(T_n z_n, Tz_n, a) + \rho(Tz_n, T_n z_n, a).$$

From (*)

$$\alpha_1 \rho(Tz_n, Tz, a) + \alpha_2 \rho(z_n, Tz_n, a) + \alpha_3 \rho(z, Tz, a) - \min\{\rho(z_n, Tz, a), \ \rho(z, Tz_n, a)\} \leq \beta \rho(z_n, z, a).$$

As in the proof of Theorem 3.4, we have two possibilities:

$$\rho(Tz_n, Tz, a) \leq \left(\frac{\beta+1}{\alpha_1}\right) \rho(z_n, z, a) - \frac{\alpha_2}{\alpha_1} \rho(z_n, Tz_n, a)$$

or

$$\rho(Tz_n, Tz, a) \leq \left(\frac{\beta}{\alpha_1 - 1}\right) \rho(z_n, z, a) - \left(\frac{\alpha_2}{\alpha_1 - 1}\right) \rho(z_n, Tz_n, a).$$

Using either of the two cases, we get

$$\rho(T_n z_n, Tz, Tz_n) = \rho(Tz, Tz_n, z_n) = 0.$$

Hence one obtains

$$\rho(z_n, z, a) \leq \rho(T_n z_n, Tz_n, a) + \left(\frac{\beta+1}{\alpha_1}\right) \rho(z_n, z, a) - \frac{\alpha_2}{\alpha_1} \rho(T_n z_n, Tz_n, a)$$

or

$$\rho(z_n, z, a) \leq \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 - \beta - 1}\right) \rho(T_n z_n, Tz_n, a).$$

Here right hand side approaches to zero as $n \to \infty$ giving thereby $z_n \to z$.

Similarly using second possibility we can conclude that $z_n \to z$. This ends the proof.

ACKNOWLEDGEMENT

I am grateful to Proof. K. Iséki for providing me with a reprint of this paper [6].

REFERENCES

[1] Lj. B. Ćirić, *On some maps with a non-unique fixed point*, Publ. Inst. Math. 7 (31) (1974), 52—58.

[2] S. Gähler, *2-metrische Raüme und ihre topologische struktur*, Math. Nachr. 26 (1963/64) 115—148.

[3] S. Gähler, *Linear 2-normierte Raüme*, Math. Nachr. 28 (1965), 1—43.

[4] S. Gähler, *Über 2-Banach-Raüme*, Math. Nachr. 42 (1969), 335—347.

[5] K. Iséki, *A property of orbitally continuous mappings in 2-metric spaces*, Math. Seminar Notes XVIII, Kobe Univ. (1975).

[6] K. Iséki, *Fixed point theorems in 2-metric spaces*, Math. Seminar Notes XIX. 1975.

[7] K. Iséki, P. L. Sharma and B. K. Sharma, *Contraction type mapping on 2-metric space*, Math. Japon. 21 (1976), 67—70.

[8] M. R. Tasković, *Some results in the fixed point theory*, Publ. Inst. Math. 20 (34) (1976), 231—242.

[9] A. G. White Jr, *2-Banach spaces*, Math. Nachr. 42 (1969), 43—60.

Department of Pure Mathematics,
La Trobe University,
Bundoora, Victoria, 3083,
Australia.