PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle sér.e, tome 27 (41) 1980, pp. 103—106

ON A MANY-VALUED SENTENTIAL CALCULUS

Miodrag Kapetanović

Tableau method of Smullyan (see [2]), based on the work of Gentzen, can be extended to a class of many-valued sentential logics in the following way.

Let $(E, \leqslant)$ be a chain with the least element 0 and the largest element 1 $(\neq 0)$. For every $e \in E$ define $j_e : E \to E$ as

$$j_e(x) = \begin{cases} 1, & \text{if} \quad x = e \\ 0, & \text{if} \quad x \neq e. \end{cases}$$

Let us also fix $s \in E$ such that $0 < s \leqslant 1$ and define $\sim$ as

$$\sim x = \begin{cases} 1, & \text{if} \quad x < s \\ 0, & \text{if} \quad x \geqslant s. \end{cases}$$

To the structure $\mathfrak{E} = (E, \ldots j_e \ldots (e \in E), \sim, \max, \min)$[1] we then adjoin a language of the same type i. e. one with unary connectives J_e (for every $e \in E$) and $\neg$ and binary connectives $\vee$, $\wedge$. The set For of all formulae of the language is defined in a standard way, supposing that denumerably many sentence letters are available. Thus $\mathfrak{E}$ is a valuation system for this language (in the sense of [1]) and $v : \text{For} \to E$ is called a valuation iff for all formulae A, B and all $e \in E : v(J_e(A)) = j_e(v(A))$, $v(\neg A) = \sim v(A)$, $v(A \vee B) = \max(v(A), v(B))$, $v(A \& B) = \min(v(A), v(B))$. A formula A is *satisfied* by a valuation v iff $v(A) \geqslant s$. A set of formulae is satisfied by v iff v satisfies all its members. A formula is a *tautology* iff it is satisfied by all valuations.

Call π a *prefix* iff for some $e \in E$ it is one of the symbols e, $<e$, $\leqslant e$, $\geqslant e$, $>e$. Then if A is a formula and π a prefix, the pair (π, A) is a *prefixed formula*, denoted by πA.

---

[1] Max and min are binary operations on E defined as usual, so that a sort of implication can be defined as well, by putting $x \supset y = \max(\sim x, y)$.

The following list defines components of prefixed formulae:

formula	*its components*
$\pi_0 J_{e'}(A)$	$\{e' A\}$
$\pi_1 \neg A$	$\{\geqslant sA\}$
$\pi_0 \neg A$	$\{<sA\}$
$\pi_1 A \vee B$	$\{\pi_1 A,\ \pi_1 B\}$
$\pi_0 A \& B$	$\{\pi_0 A,\ \pi_0 B\}$
$\pi_1 J_{e'}(A)$	$\{<e' A\},\ \{>e' A\}$
$\pi_0 A \vee B$	$\{\pi_0 A\},\ \{\pi_0 B\}$
$\pi_1 A \& B$	$\{\pi_1 A\},\ \{\pi_1 B\}$
$eA \vee B$	$\{eA,\ \leqslant eB\},\ \{<eA,\ eB\}$
$eA \& B$	$\{eA,\ \geqslant eB\},\ \{>eA,\ eB\}$.

Here $e,\ e' \in E$, $\pi_0 \in \{\geqslant e,\ >e\}$, $\pi_1 \in \{<e,\ \leqslant e\}$ and $\pi_0,\ \pi_1 \notin \{>0,\ \leqslant 1,\ <0.\ >1\}$.

In order to make use of Smullyan's elegant unified notation (and using the above symbolism) we denote $\pi_0 J_{e'}(A)$, $\pi_1 \neg A$, $\pi_0 \neg A$, $\pi_1 A \vee B$, $\pi_0 A \& B$ by α and its components by α_1, as well as $\pi_1 J_{e'}(A)$, $\pi_0 A \vee B$, $\pi_1 A \& B$, $eA \vee B$, $eA \& B$ by β and its componen s by β_1, β_2.

We can now introduce the main notion, namely that of a *tableau for a formula*. We define it (for a given formula A) by the following (and no other) inductive rules:

1° $\{\{<sA\}\}$ is a tableau for A;

2° if $\mathcal{T} \cup \{S \cup \{\alpha\}\}$ is a tableau for A, then $\mathcal{T} \cup \{S \cup \alpha_1\}$ is a tableau for A;

3° if $\mathcal{T} \cup \{S \cup \{\beta\}\}$ is a tableau for A, then $\mathcal{T} \cup \{S \cup \beta_1,\ S \cup \beta_2\}$ is a tableau for A.

Elements of a tableau are called *branches*. A branch S is *closed* iff for some formula A one of the following hold:

a) $<0\,A$ or $>1\,A$ are in S;

b) for some $e,\ e' \in E$ such that $e \neq 0,\ 1$, $e \neg A$ or $eJ_{e'}(A)$ are in S;

c) for some $e \in E$ one of $\{>eA,\ eA\}$, $\{<eA,\ \geqslant eA\}$, $\{<eA,\ >eA\}$, $\{eA.\ >eA\}$ is a subset of S;

d) for some $e,\ e' \in E$ such that $e < e'$, one of $\{\pi A,\ \pi' A\}$, where

$$\pi \in \{<e,\ \leqslant e,\ e\}$$

and

$$\pi' \in \{e',\ \geqslant e',\ >e'\},$$

is a subset of S.

A branch is *open* iff it is not closed. A tableau is closed iff all its branches are closed, otherwise it is open. Call A a *theorem* iff there is a closed tableau for A. Thus the tableau method is a way of proving by showing the impossibility of a refutation.

For our basic results we first need one more definition. Say that eA is *v-true* iff $v(A)=e$; also $\geqslant eA$ is v-true iff $v(A)\in\{e'\in E \mid e'\geqslant e\}$ and similarly for other three forms of prefixes. A set of prefixed formulae is v-true iff all its members are. Now it is easy to prove, considering all cases from the list, the following.

Lemma 1. *For all valuations v, α is v-true iff α_1 is, and β is v-true iff at least one of β_1, β_2 is v-true.*

For instance $\geqslant eJ_{e'}(A)$ is v-true iff $v(J_{e'}(A))\in\{e'' \mid e''\geqslant e'\}$ iff $v(J_{e'}(A))=1$ iff $v(A)=e'$ iff $e'A$ is v-true iff $\{e'A\}$ is v-true. Other cases are similar.

It is also easy to check that if some closure condition holds of a branch S, then there is no v such that all formulae in S are v-true. This implies.

Lemma 2. *If a branch is v-true for some v, then it is open.*

Our first basic result is.

Consistency lemma. *Every theorem is a tautology.*

Proof. If A is not a tautology then $<sA$ is v-true for some v, i. e. the tableau $\{\{<sA\}\}$ has a v-true branch. Suppose $S\cup\{\alpha\}$ is a v-true branch of a tableau for A. Then $S\cup\alpha_1$ is also v-true by Lemma 1. The case β is similar, so it follows that all tableaux for A have a v-true branch. But these branches are open by Lemma 2, hence no tableau for A is closed i. e. A is not a theorem.

The converse of this result is also true and it is called.

Hintikka's lemma *Every tautology is a theorem.*

Proof. Given a formula A, there is a maximal tableau for A i. e. one without α'*s* and β'*s* and if A is not a theorem then it contains an open branch S_0. Define a valuation v as follows:

$$v(p)=\begin{cases}\text{some } e\in\bigcap\{\pi \mid \pi p\in S_0\}, & \text{if the sentence letter } p \text{ ocurrs in } S_0\\ 0, & \text{otherwise}\end{cases}$$

(here, for the sake of simplicity, π is used to denote both a prefix and its corresponding subset of E). The above intersection is certainly nonempty since S_0 is open i. e. no closure condition applies. It follows that the whole of S_0 is v-true since all $\pi p\in S_0$ are v-true by definition of v and other elements of S_0 are of the form $\geqslant 0\,A$, $\leqslant 1\,A$, so obviously v-true. Now suppose S_0 was built from a branch $S_1=T\cup\{\alpha\}$ by putting $S_0=T\cup\alpha_1$. Then S_1 is v-true by Lemma 1. Next if S_0 was built from $S_1=T\cup\{\beta\}$ by putting $S_0=T\cup\beta_1$ or $S_0=T\cup\beta_2$, then again S_1 v-true by Lemma 1. Similarly we find S_2 which also must be true and repeat the whole argument until, after finitely many steps, $\{<sA\}$ is reached. This must happen since the whole process is obviously well-founded and so $<sA$ is v-true for some v, i. e. A is not a tautology.

Finally, note that for all formulae A, the following are finite: tableaux for A, the set of all tableaux for A, the number of all $e \in E$ occuring in a tableau for A, the set of prefixes in a tableau for A, the number of sorts of prefixes and the number of closure conditions. This indicates that the testing of A takes only finitely many steps, i.e. we get the following.

Corollary. *The set of all tautologies is decidable.*

REFERENCES

[1] M. Dummet, *Elements of Intuitionism*, Clarendon Press, Oxford 1977.

[2] R. M. Smullyan, *First-Order Logic*, Springer-Verlag, Berlin Heidelberg New York, 1968.

Miodrag Kapetanović
Matematički institut, Beograd