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ULTRAPRODUCTS OF WELL ORDERS

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In this paper the structure of ultraproducts of well ordered structures is considered. The notions of filter regularity, uniformitv, completness, descending completness, decomposition, good ultrafilter and other are as usual (like in (2), (4), (5)). In the following text D is an uniform ultrafilter over a cardinal k, cardinals are denoted with λ, μ, ν, ordinals with α, β, γ, ξ, η.

L e m m a 1. Let, for $i\in k$, A_i be well ordered by $<_i$ and let

$$\langle A, <_A\rangle = \prod_D \langle A_i, <_i\rangle.$$

Then for all $f_D\in A$

$$\{g_D\in A \mid g_D<_A f_D\}\cong\prod_D \langle [f(i)], <_i\rangle,$$

where $[f(i)]$ is the set of predecessos of $f(i)$ in A_i.

C o r o l l a r y 2. Let $\operatorname{esssup}_D\{\lambda_\xi/\xi<k\}=\lambda$, then $\prod_D \langle\lambda_\xi, <\rangle_{\xi\in k}$ is isomorphic to some initial segment of $\prod_D \langle\lambda, <\rangle$.

Hence, without any essential loss we can consider just the following case

$$\langle A, <\rangle = \prod_D \langle\lambda, <\rangle, \quad \lambda\geqslant k.$$

Since, for $\tau<\sigma$, $\prod_D \langle\tau, <\rangle$ is isomorphic to some initial segment of $\prod_D \langle\sigma, <\rangle$, we can identify $\prod_D \langle\tau, <\rangle$ with that initial segment and accept $\prod_D \tau\subseteq\prod_D \sigma$.

Define for $\alpha\leqslant\lambda$ the family

$$F_\alpha=(\prod \alpha)\setminus\bigcup_{\xi\in\alpha} F_\xi.$$

For all $\alpha<\lambda$, $F_{\alpha+1}=\{C_\beta\}$. Possibly more interesting case would be when α is a limit ordinal, which is assumed from now on when we write α in F_α.

If D is countably complete, then the situation is simple:

P1. If D is countably complete then for all (limit) $\alpha\leqslant\lambda$ if $cf\,\alpha\neq k$ then $F_\alpha=\varnothing$, F_k is well ordered and $2^k<ot(F_k)<(2^k)^+$. Thus, $<$ is well ordering.

Proof. using theorems 4.2.13. and 4.2.21. of (2).

P2. For all $\alpha\leqslant\lambda$ and $f_D\in F_\alpha$, $|(\cdot, f_D)_{F_\alpha}|\leqslant|(f_D, \cdot)_{F_\alpha}|$ ie

$$|\{g_D\in F_\alpha\,|\,g<_D f\}|\leqslant|\{g_D\in F_\alpha\,|\,g\underset{D}{>}f\}|.$$

Proof. translating $(\cdot, f_D)_{F_\alpha})$ with f_D:

$$f_D+(\cdot, f_D)_{F_\alpha}=\{(f+g)_D\,|\,g_D\in(\cdot, f_D)_{F_\alpha}\}\subseteq(f_D, \cdot)_{F_\alpha},$$

since α is a limit ordinal and $f_D+(\cdot, f_D)_{F_\alpha}\cong(\cdot, f_D)_{F_\alpha}$.

P3. For all α, $F_{cf\alpha}=\varnothing$ iff $F_\alpha=$, and $|F_{cf\alpha}|\leqslant|F_\alpha|$.

Proof. let $\nu=cf\,\alpha$ and $\alpha=\bigcup\limits_{\xi<\nu}\alpha_\xi$, α_ξ being increassing. For $f_D\in F_\nu$ define $\bar{f}$ by $\bar{f}(i)=\alpha_\xi$ iff $f(i)=\xi\cdot\bar{f}_D\in F_\alpha$, for otherwise we would have $\bar{f}_D\in F_{\alpha_\xi}$, hence $f_D\in F_\xi$, contradicting $f_D\in F_\nu$. Other way. For $f_D\in F_\alpha$ define $f':k\to\nu\,(=cf\,\alpha)$, by $f'(i)=\xi_i$ iff $\xi_i=\min\{\varkappa\in\nu\,|\,f(i)\in\alpha_\varkappa\}$. If $f_D'\notin F_\nu$ then for some $\xi<\nu$, $f_D'\in F_\xi$ ie $f'<\xi$ A.E., so $f_D\in F_{\alpha_\xi}$ contradicting $f_D\in F_\alpha$.

P4. If $\alpha<\beta$ and $cf\,\alpha=cf\,\beta$ then $|F_\alpha|\leqslant|F_\beta|$.

Proof. similar to P3.

Proposition 3. (from (3)). Let D be a countably incomplete ultrafilter. Then for all A_i, $|\prod\limits_D A_i|$ is either finite or $\geqslant 2^\omega$.

P5. For all α $st\,cf\,\alpha=\omega$ and all $f_D\in F_\alpha$

$$|F_\alpha|\geqslant|(\cdot, f_D)_{F_\alpha}|\geqslant 2^\omega.$$

Proof. using Lemma 1, Proposition 3, P2 and P3.

P6. $F_\mu\neq\varnothing$ implies $|F_\mu|\geqslant\sum\limits_{\alpha<\mu}|F_\alpha|$ and $|F_\mu|=|\prod\limits_D\mu|$.

Proof. given $f_D\in F_\mu$, for any α, we have $f_D+F_\alpha\subseteq F_\mu$.

If $g<_D f$ and $g\in\prod\limits_D\mu$ then $(f+g)_D\in F_\mu$. Now use P2.

P7. $F_{|\alpha|}\neq\varnothing$ implies $|F_{|\alpha|}|\geqslant|F_\alpha|$.

Theorem 4. *(from* (1)) *D is ν-descendingly complete iff $d(\nu)$ is cofinal in* $\prod\limits_D\langle\alpha, <\rangle$.

Theorem 5. *(from (3)) If D is ν-regular then D is ν-descendingly incomplete.*

P8. $F_\alpha = \varnothing$ iff D is $cf\alpha$-descendingly complete or $cf\alpha > k$.

Proof. using Theorem 4, P3. and the fact that for $cf\alpha > k$ every function in $\prod_D \alpha$ is bounded with some constant.

P9. If D is μ-regular then for all $\alpha \leqslant \mu$, $|_{F\alpha}| \geqslant 2^\mu$.

Proof. for all $\nu < \mu$, D is ν-regular and hence by Theorem 5, ν-descendingly incomplete. From P8 it follows that for all $\alpha \leqslant \mu$ F_α is nonempty. Theorem 4.3.7. of (2) and P6. give the inequality.

Theorem 6. *(from (5)) If μ is regular then every μ-descenddingly complete ultrafilter is $\mu^{\pm}$ desceningly complete,*

Crollary 7. If $k = \omega_n$ then for all m$\leqslant$n, D is not ω_m-descendingly complete.

P10. If $k = \omega_n$ then for all $\alpha \leqslant k$, $|F_\alpha| \geqslant 2^\omega$.

P11. If for some ν, $F_\nu = \varnothing$ then for al n, $k > \nu^{(n)}$.

If k is real valued measurable cardinal then as J. Silver has proved any ultrafilter extending the filter of sets of measure 1 is not μ regular for any $\mu \neq \omega, k$. It can he checked that every such ultrafilter is not (μ, μ) regular for any μ such that $cf\mu \neq \omega, k$. Hence for all α if $cf\alpha \neq \omega, \mu$ then D is not $cf\alpha$-descendingly complete, hence.

P12. if D is as above then for all α, $cf\alpha \neq \omega, k$ implies $F_\alpha = \varnothing$.

Definition. We say that D is μ-weakly normal iff $\prod_D \mu$ has a minimal unbonded (ie not bounded by constant) function. It is known that D is k-weakly normal iff there is some weakly normal ultrafilteɹ below D in the Rudin--Keisler order.

We do not know the answers on the following questions.

Q1. Are there D and μ such that D is μ-weakly normal uniform ultrafilter over some $k > \mu$?

Q2. Are there D and $\alpha \neq k$ and $f_D \in F_\alpha$ such that

$$|(\cdot, f_D)_{F\alpha}| < |(f_D, \cdot)_{F\alpha}|?$$

Q3. Are there D and α such that $F_\alpha \neq \varnothing$ and $|F_{|\alpha|}| > |F_\alpha|$?

Q4. Are there D, α, β such that $\alpha < \beta$, F_α, $F_\beta \neq \varnothing$ and $|F_\alpha| > |F_\beta|$?

Q5. Are there D and μ such that D is (μ, μ)-regular and $F_\mu = \varnothing$?

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