PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle, série, tome 27 (41) 1980, pp. 91—98

ON CERTAIN SUMS INVOLVING VON MANGOLDT'S FUNCTION IN SHORT INTERVALS

Aleksandar Ivić

(Received April 15. 1979)

1. Introduction

Let r denote positive integers with the property that

$$x \log^{-a} x \ll \sum_{x<r\leq 2x} 1 \ll x \log^{-a} x \tag{1.1}$$

for some fixed $a \geq 0$ and some absolute constants implied by the symbol $\ll$. Here as usual $f(x) \ll g(x)$ is equivalent to $f(x)=O(g(x))$ and means $|f(x)| \leq \leq Cg(x)$ for some fixed $C>0$ and $x \geq x_0$. The aim of this note is to prove as $x \to \infty$

$$\sum_{Q<r\leq 2Q} \left(\psi\left(\frac{x+h}{r}\right) - \psi\left(\frac{x}{r}\right)\right) = (1+o(1))\, h \sum_{Q<r\leq 2Q} 1/r, \tag{1.2}$$

where Q and h are suitably chosen functions of x, and to deduce hence the existence of integers of the form pr in the interval $(x, x+h]$ for x sufficiently large. Here as usual p denotes a prime and

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = \sum_{p^\alpha \leq x} \log p, \tag{1.3}$$

where the von Mangoldt function $\Lambda(n)$ is defined to be $\log p$ if $n=p^\alpha$ and zero otherwise. Since no other properties of the integers r are used besides the weak estimate (1.1), it is natural to expect that sharper results than ours may be obtained by utilizing deeper properties of each particular sequence r. The principal tool in dealing with the left-hand side of (1.2) is E. Landau's classical formula ([6], Ch. 7)

$$\psi(x) = x - \sum_{|\gamma|\leq T} x^\rho/\rho + O(xT^{-1}\log^2 xT) + O(\log x), \tag{1.4}$$

where $T=T(x)$ will be suitably chosen later, and $\rho=\beta+i\gamma$ is a zero of the zeta function $\zeta(s)$ with $0\le\beta\le 1$ and $-T\le\gamma\le T$. As usual $N(\sigma, T)$ will denote the number of zeros ρ satisfying $0\le\sigma\le\beta\le 1$ and $-T\le\gamma\le T$. Estimates for $N(\sigma, T)$ may be written as

(1.5) $$N(\sigma, T)\ll T^{A(\sigma)(1-\sigma)}\log^D T,$$

where we suppose that the $\ll$-constant is uniform in σ. For $\sigma\le 1/2$ we have trivially $A(\sigma)(1-\sigma)=1$, $D=1$, while for $\sigma\ge 1/2$ we have $A(\sigma)(1-\sigma)\le 1$ and $A(\sigma)(1-\sigma)$ is nonincreasing. Our main result will be the following

Theorem. Let r denote positive integers satisfying (1.1). Suppose $C>2$ is the number such that (1.5) holds with $A(\sigma)\le C$ uniformaly in σ, and that further $A(\sigma)\le C_1<C$ with some $C_1\ge 2$ uniformly for $u\le\sigma\le 1$, where u is a number satisfying $1/2<u<1$. Then if $Q=x/h$, $h\ge x^{1-(C/2+1)^{-1}}\log^M x$, we have as $x\to\infty$

$$\sum_{Q<r\le 2Q}\left(\psi\left(\frac{x+h}{r}\right)-\psi\left(\frac{x}{r}\right)\right)=(1+o(1))\,h\sum_{Q<r\le 2Q}1/r,$$

provided that

(1.6) $$M>(D+2a+9)/(C+2)(1-u).$$

2. Proof of the theorem

Supposing $T=T(x)\le x$ we obtain from (1.4)

(2.1) $$\sum_{Q<r\le 2Q}\left(\psi\left(\frac{x+h}{r}\right)-\psi\left(\frac{x}{r}\right)\right)=$$

$$=h\sum_{Q<r\le 2Q}1/r-S+O\left(xT^{-1}\log^2 x\cdot\sum_{Q<r\le 2Q}1/r\right),$$

where we have set

(2.2) $$S=\sum_{|\gamma|\le T}x^\rho C(x,\rho)P_Q(\rho),$$

(2.3) $$C(x,\rho)=((1+h/x)^\rho-1)/\rho,$$

(2.4) $$P_Q(\rho)=\sum_{Q<r\le 2Q}r^{-\rho}.$$

To estimate various sums involving zeros of $\zeta(s)$ we shall use ($h=o(x)$ as $x\to\infty$)

$$((x+h)^\rho-x^\rho)/\rho=\int_x^{x+h}z^{\rho-1}\,dz\ll\int_x^{x+h}|z^{\rho-1}|\,dz\ll hx^{\beta-1},$$

which yields

(2.5) $$C(x,\rho)\ll\min(1/Q,\ 1/|\gamma|),$$

where

(2.6) $$Q=x/h.$$

We shall also use the following inequality (A. Walfisz [8]) for the zero-free region of $\zeta(s)$:

$$\beta \leq 1 - K\delta(|\gamma|), \tag{2.7}$$

where $|\gamma| \geq \gamma_0$, $\delta(x) = \log^{-2/3} x \cdot (\log\log x)^{-1/3}$, $K>0$, so that $x^{-K\delta(x)} \ll \log^{-A} x$ for any fixed $A>0$ and $K>0$ (in what follows A and K may denote different positive, absolute constants). We now choose

$$h = x^{1-(C/2+1)^{-1}} \log^M x, \quad T = x^{(C/2+1)^{-1}} \log^{2-M} x \cdot \log\log x, \tag{2.8}$$

where C and M are numbers that appear in the formulation of the theorem. Since $Q<T$ by (2.6), we may write S as

$$S = S_1 + S_2 + S_3 + S_4, \tag{2.9}$$

where using (2.5) we obtain

$$S_1 \ll Q^{-1} \sum_{\beta \geq u,\, |\gamma| \leq Q} x^\beta |P_Q(\rho)|, \tag{2.10}$$

$$S_2 \ll \sum_{\beta \geq u,\, Q < |\gamma| \leq T} x^\beta |P_Q(\rho)| / |\gamma|, \tag{2.11}$$

$$S_3 \ll Q^{-1} \sum_{0<\beta<u,\, |\gamma| \leq Q} x^\beta |P_Q(\rho)|, \tag{2.12}$$

$$S_4 \ll \sum_{0<\beta<u,\, Q<|\gamma| \leq T} x^\beta |P_Q(\rho)| / |\gamma|, \tag{2.13}$$

where u is the number that appears in the formulation of the theorem. We now introduce the weighted density function $W(\sigma, t)$ as

$$W(\sigma, t) = \sum_{\beta \geq \sigma,\, |\gamma| \leq t} |P_Q(\rho)|, \tag{2.14}$$

and proceed to estimate S_1. We have

$$\begin{aligned} \sum_{\beta \geq u,\, |\gamma| \leq Q} (x^\beta - x^u) |P_Q(\rho)| &= \log x \cdot \sum_{\beta \geq u,\, |\gamma| \leq Q} |P_Q(\rho)| \int_u^\beta x^\sigma \, d\sigma = \\ &= \log x \cdot \int_u^1 \sum_{\beta \geq \sigma,\, |\gamma| \leq Q} |P_Q(\rho)| \, x^\sigma \, d\sigma, \end{aligned} \tag{2.15}$$

which implies

$$S_1 \ll \log x \cdot \max_{u \leq \sigma \leq 1} x^\sigma \, W(\sigma, Q)/Q. \tag{2.16}$$

Similarly it follows using (2.15)

$$\sum_{\beta\geq u,\, Q<|\gamma|\leq T} x^{\beta}\,|P_Q(\rho)|\,(1/|\gamma|-1/T)=$$

(2.17)

$$\sum_{\beta\geq u,\, Q<|\gamma|\leq T} |P_Q(\rho)|\,x^{\beta}\int_{|\gamma|}^{T} t^{-2}\,dt=\int_{Q}^{T} t^{-2}\sum_{\beta\geq u,\, Q<|\gamma|\leq t} x^{\beta}\,|P_Q(\rho)|\,dt=$$

$$=\int_{Q}^{T} t^{-2}x^{u}(W(u,\,t)-W(u,\,Q))\,dt+\int_{Q}^{T} t^{-2}\log x\cdot\int_{u}^{1} x^{\sigma}(W(\sigma,t)-$$

$$-W(\sigma,\,Q))\,d\sigma\,dt\ll x^{u}\log T\cdot\max_{Q\leq t\leq T} W(u,\,t)/t+$$

$$+\log x\cdot\log T\max_{u\leq\sigma\leq 1} x^{\sigma}\max_{Q\leq t\leq T} W(\sigma,\,t)/t.$$

Therefore we obtain

$$S_1+S_2\ll\log^2 x\cdot\max_{u\leq\sigma\leq 1} x^{\sigma}\max_{Q\leq t\leq T} W(\sigma,\,t)/t. \tag{2.18}$$

The same technique leads also to

$$S_3+S_4\ll\log^2 x\cdot\max_{0\leq\sigma\leq 1} x^{\sigma}\max_{Q\leq t\leq T} W(\sigma,\,t)/t, \tag{2.19}$$

and thus we are left with estimating $W(\sigma,\,t)$. To do this we shall use the following inequality ([2], eq. (19.24) with $q=1$):

Let $s_r=\sigma_r+it_r\,(1\leq r\leq R)$ be complex numbers satisfying $0<\sigma\leq\sigma_r\leq 1$, $-T\leq t_r\leq T$, $t_{r+1}-t_r\geq\delta>0$. Then for arbitrary complex numbers a_n $(1\leq n\leq N)$ we have

$$\sum_{r\leq R}\Big|\sum_{n\leq N} a_n n^{-s_r}\Big|^2\ll(\delta^{-1}+\log N)\log N\cdot\sum_{n\leq N}(n+T)\,|a_n|^2 n^{-2\sigma} \tag{2.20}$$

Using the Cauchy-Schwarz inequality we obtain from (2.14) and (2.4)

$$W^2(\sigma,\,t)\leq N(\sigma,\,t)\sum_{\beta\geq\sigma,\,|\gamma|\leq t}\Big|\sum_{Q<r\leq 2Q} r^{-\rho}\Big|^2. \tag{2.21}$$

We now apply (2.20) to the above sum by taking $T=t$, $N=2\,[Q]$, $a_n=1$ if $n=r$ and $N/2<n\leq N$ and $a_n=0$ otherwise, and $\delta=1$ by picking representative zeros $\rho_r=\beta_r+i\gamma_r$ with $\gamma_{r+1}-\gamma_r\geq 1$, such that these zeros contain a proportion of at least $\gg 1/\log t$ zeros of all zeros counted by $N(\sigma,\,t)$. This may be done since $N(\sigma,\,t+1)-N(\sigma,\,t)\ll\log t$ uniformly in σ (see [2], Ch. 12). Hence

$$W^2(\sigma,\,t)\ll N(\sigma,\,t)\log^3 x\cdot(Q+t)\,Q^{1-2\sigma},$$

and since $t\leq T=Q\log^2 x\cdot\log\log x$ by (2.6) and (2.8), this implies

$$W(\sigma,\,t)\ll(N(\sigma,\,t))^{1/2}\,Q^{1-\sigma}\log^{5/2} x\cdot\log\log x. \tag{2.22}$$

For $u \leq \sigma \leq 1$ we have supposed $A(\sigma) \leq C_1 < C$, and so using (2.7), (2.8), (2.18) and (2.22) we obtain

$$S_1 + S_2 \ll \log^A x \cdot \max_{u \leq \sigma \leq 1 - K\delta(x)} x^\sigma Q^{A(\sigma)(1-\sigma)/2-1} Q^{1-\sigma} \ll$$

$$h \log^A x \cdot \max_{u \leq \sigma \leq 1-K\delta(x)} x^{\sigma-1} Q^{(C_1/2+1)(1-\sigma)} \ll \tag{2.23}$$

$$h \log^A x \cdot x^{-K(C-C_1)\delta(x)/(C+2)} = o(h \log^{-a} x)$$

since $C > C_1$. Similarly we obtain from (2.19)

$$S_3 + S_4 \ll \log^2 x \cdot \max_{0 \leq \sigma \leq u} x^\sigma \max_{Q \leq t \leq T} t^{A(\sigma)(1-\sigma)/2-1} \log^{D/2} x \cdot \log^{5/2} x \cdot \log\log x \cdot Q^{1-\sigma}$$

$$\ll h \log^{(D+9)/2} x \cdot \log\log x \cdot \max_{0 \leq \sigma \leq u} x^{\sigma-1} Q^{(A(\sigma)/2+1)(1-\sigma)} \ll \tag{2.24}$$

$$\ll h \log^{(D+9)/2} x \log\log x \cdot \max_{0 \leq \sigma \leq u} x^{\sigma-1} (x^{(C/2+1)^{-1}} \log^{-M} x)^{(C/2+1)(1-\sigma)} \ll$$

$$\ll h \log\log x \cdot (\log x)^{((D+9)/2 - M(1-u)(C/2+1))} = o(h \log^{-a} x),$$

since by hypothesis (1.6) holds.

This means that $S = o(h \log^{-a} x)$ with h and T given by (2.8). Finally by (1.1) we have

$$\log^{-a} x \ll \sum_{Q < r \leq 2Q} 1/r \ll \log^{-a} x,$$

and this implies by (2.1) that (1.2) holds, since by our choice of T we have

$$xT^{-1} \log^2 x \cdot \sum_{Q < r \leq 2Q} 1/r \ll h (\log\log x)^{-1} \sum_{Q < r \leq 2Q} 1/r = o(h \log^{-a} x).$$

3. Applications and remarks

Before we proceed to give some applications of our theorem, it should be remarked that our result is a generalization of an unpublished result of D. Wolke [9], who has proved Cor. 2 with $h = x^{6/11+\varepsilon}$. Both proofs utilize (2.20) (which may be regarded as a a mean-value theorem for Dirichlet polynomials) to estimate the weighted density function $W(\sigma, t)$. The same idea was used by H. Iwaniec and M. Jutila [5], where by a combination of sieve and analytic methods they prove that $p_{n+1} - p_n \ll p_n^{13/23+\varepsilon}$, where p_n denotes the n-th prime number.

To deduce number-theoretic corollaries from our theorem we may write

$$\sum_{Q < r \leq 2Q} \left(\psi\left(\frac{x+h}{r}\right) - \psi\left(\frac{x}{r}\right) \right) = \sum_{Q < r \leq 2Q} \sum_{x/r < p \leq (x+h)/r} \log p + R, \tag{3.1}$$

where

$$R = \sum_{Q<r\leq 2Q} \sum_{x/r<p^{\alpha}\leq(x+h)/r,\ \alpha\geq 2} \log p \ll \tag{3.2}$$

$$\ll \sum_{Q<r\leq 2Q} (1+((x+h)/r)^{1/2}-(x/r)^{1/2}) \log^2 x \ll$$

$$Q\log^{2-a} x + hx^{-1/2}\log^2 x \cdot \sum_{Q<r\leq 2Q} r^{-1/2} \ll (Q+hQ^{1/2}x^{-1/2})\log^{2-a} x = o(h\log^{-a} x).$$

Therefore we obtain

Corollary 1. Under the hypotheses of the theorem we have for $x\geq x_0$

$$\sum_{x<pr\leq x+h,\ Q<r\leq 2Q} 1 \gg h\log^{-1-a} x, \tag{3.3}$$

so that there is an integer of the form pr in the interval $(x, x+h]$ for $x\geq x_0$ and $h\geq x^{1-(C/2+1)^{-1}}\log^M x$, where M satisfies (1.6).

Using the explicit formula for the sum $\psi(x; k, l) = \sum_{n\leq x,\ n\equiv l \pmod k} \Lambda(n)$ instead of (1.4) (see [6], Ch. 9, eq. (2.5)), one could replace (3.3) with

$$\sum_{x<pr\leq x+h,\ p\equiv l \pmod k} 1 \gg h\log^{-1-a} x, \tag{3.4}$$

where k and l are fixed coprime integers, and h and M are as in the theorem.

Note that for any fixed $\varepsilon>0$ we may take

$$C=12/5,\ D=9,\ u=3/4+\varepsilon. \tag{3.5}$$

This follows from the estimates of A. E. Ingham and M. N. Huxley (see [2] and [3])

$$N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)} \log^5 T, \tag{3.6}$$

$$N(\sigma, T) \ll T^{(5\sigma-3)(1-\sigma)/(\sigma^2+\sigma-1)} \log^9 T, \tag{3.7}$$

Since $3/(2-\sigma)\leq 12/5$ for $\sigma\leq 3/4$, and $(5\sigma-3)/(\sigma^2+\sigma-1)$ is decreasing for $\sigma\geq 3/4$ with a maximum of $12/5$ at $\sigma=3/4$. Thus $A(\sigma)<12/5-\varepsilon_1$ for $\sigma\geq 3/4+\varepsilon$, where ε_1 is a positive number depending on ε.

If we choose now for r the sequence of primes, then $a=1$ in (1.1), and from Cor. 1 and (3.5) we obtain

Corollary 2. For $x\geq x_0$, $Q=x/h$ and $h\geq x^{6/11}\log^{19} x$ we have

$$\sum_{x<p_1p_2\leq x+h,\ Q<p_1\leq 2Q} 1 \gg h\log^{-2} x, \tag{3.8}$$

where p_1, p_2 denote primes.

This result is almost as good as H. — E. Richert's sieve result ([1], Ch. 9), but the author has been kindly informed by D. Wolke that H. Halberstam, D. R. Heath-Brown and H. — E. Richert have recently jointly proved that there is a $p_1 p_2$ in $(x, x+x^w]$ for $x \geq x_0$ and $w = 0.45\ldots$. Sieve techniques, however, have the disadvantage of rarely being able to produce an asymptotic formula like (1.2), but only a lower bound of the right order of magnitude.

As another example, let r denote now integers such that both r and $r+1$ are representable as a sum of two integer squares (such numbers are called B-twins). It follows from the work of K. — H. Indlekofer [1] that (1.1) holds with $a=1$ if r denotes B-twins. Therefore similarly as in the previous example we obtain

Corollary 3. If $x \geq x_0$ an $h \geq x^{6/11} \log^{19} x$, then

$$\sum_{x < pr \leq x,\ r \in R} 1 \gg h \log^{-2} x \tag{3.9}$$

where R denotes the set of B-twins.

Finally let r denote integers representable as

$$r = p_1{}^2 + p_2{}^2, \tag{3.10}$$

where p_1 and p_2 are odd primes. G. J. Rieger proved in [7] the asymptotic formula

$$\sum_{r \leq x} 1 = \frac{\pi}{2}\, x \log^{-2} x \cdot (1 + O(\log^{-2/3} x \cdot \log x)^{2/3})), \tag{3.11}$$

so that in this case (1.1) holds with $a=2$, and we obtain

Corollary 4. If $x \geq x_0$ and $h \geq x^{6/11} \log^{21} x$, then

$$\sum_{x < pr \leq x+h} 1 \gg h \log^{-3} x, \tag{3.12}$$

where r denotes integers of the from (3.10).

Finally I wish to thak D. Wolke for making his unpublished manuscript [9] available to me, and Mathematical Inst. of Belgrade and Rep. Zajednica of Serbia for financing this research.

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Aleksandar Ivić
Rudarsko-geološki fakultet
Đušina 7, 11000 Beograd
Jugoslavija