PUBLIATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 27 (41), 1980, pp. 77—82

SOME EXTENSIONS OF FIXED POINT THEOREMS CONCERNIG ĆIRIĆ'S QUASI-CONTRACTION MAPPINGS

M. Hegedüs

Introduction

Let X be a metric space with distance function ρ, f a mapping of X into itself. Let consider the next conditions:

(A*) $\rho(f(x),\ f(y)) \leqq \alpha \cdot \operatorname{diam}\{x, y\}$,

(B*) $\rho(f(x),\ f(y)) \leqq \alpha \cdot \operatorname{diam}\{x, y, f(x), f(y)\}$,

(C*) $\rho(f(x),\ f(x)) \leqq \alpha \cdot \operatorname{diam}\{x, y, f(x), f(y), f^2(x), f^2(y), \ldots\}$.

and statements

(a) f has unique fixed point x^* in X,

(b) $x^* = \lim\limits_{n \to \infty} f^n(x)$ for all $x \in X$,

(c) $$\rho(x^*, f^n(x)) \leqq \frac{\alpha^n}{1-\alpha} \cdot \rho(x, f(x)) \quad (n=0, 1, \ldots),$$

(c*) $$\rho(x^*, f^n(x)) \leqq \frac{\alpha}{1-\alpha} \cdot \rho(f^{n-1}(x), f^n(x)) \quad (n=1, 2, \ldots).$$

In the fixed-point theory of contractive operators (mappings whitch shrinks distance in some manner) on metric spaces is well-known the result of the Polish matematican *S. Banach* [2]. Banach's contraction principle can be formulated as follows.

Let $f: X \to X$ be a mapping of a complete metric space (X, ρ) into itself. If f is a contraciton, i.e. if

(A) $$\rho(f(x), f(y)) \leqq \alpha \cdot \rho(x, y), \quad (0 \leqq \alpha < 1;\ x, y \in X)$$

then the statements (a), (b) (c) *and* (c*) *are true.*[1] It is obvious that the condition (A) is equivalent to (A*).

---

[1] It is easy to verify that (c) implies (c*).

In [4] *Lj. B. Ćirić* considered generalized contractions, defined as follows.

A mapping $f: X \to X$ is said to be a *quasi-contraction* if there exists a number α, $0 \leqq \alpha < 1$ such that

(B) $\rho(f(x), f(y)) \leqq \alpha \cdot \max\{\rho(x, y),\ \rho(x, f(x)),\ \rho(y, f(y)),\ \rho(x, f(y)),\ \rho(y, f(x))\}$

holds for every $x, y \in X$.

It has been shown in [4] that the condition of quasicontractivity implies all conclusions of Banach's contraction principle.

It is easy to werify that the condition (B^*) is equivalent to (B).

Special cases of (B) or (B*) have been discussed by *R. Kannan* [9], *K. Chatterjea* [5], *S. Reich* [12], *G. E. Hardy* and *T. D. Rogers* [7], *Lj. B. Ćirić* [3], *T. Zamfirescu* [15] and others. A comparative study of these generalizations has been made recently by *B. E. Rhoades* [14].

In [8] we considered generalized contractions, defined as follows.

A mapping $f: X \to X$ is said to be a *generalized qasi-contraction* if

$$\operatorname{diam}\{x, f(x), f^2(x), \ldots\} < \infty \quad \text{for all } x \in X$$

and there exists a number α, $0 \leqq \alpha < 1$ such that the condition (C*) holds for every $x, y \in X$.

We have proved in [8] that the condition of generalized quasi-contractivity implies all conclusions of Banach's contraction principle. Put

$$\rho^*(a, b) = \sup\{\rho(a, f^k(b)) : k = 0, 1, 2, \ldots\}.$$

We shall show that (C*) is equiavlent to

(C) $\rho(f(x), f(y)) \leqq \alpha \cdot \max\{\rho(x, y),\ \rho^*(x, f(x)),\ \rho^*(y, f(y)),\ \rho^*(x, f(y)),$
$\rho^*(y, f(x))\}.$

In [8] it has been shown that the generalized quasi-contractions are really extensions of *Ćirić's* quasi-contraction mappings. Recently *B. E. Rhoades* [14] and *J. Achari* [1] proved some fixed-point theorems concerning *Ćirić's* quasi-contractions. In this paper we extend their results for generalized quasi-contractions.

1. Lemmas

Let $\{x_n\}_{n=0}^{\infty}$ be a bounded sequence of elements of X. Put

$$\delta_n = \operatorname{diam}\{x_n, x_{n+1}, \ldots\} \qquad (n = 0, 1, \ldots).$$

The first of this lemma is fundamental

Lemma 1. *Suppose that there exists a number q, $0 \leqq q < 1$ such that*

(1) $$\delta_{n+1} \leqq q \cdot \delta_n, \qquad (n = 0, 1, \ldots).$$

Then

(2) $$\delta_n = \sup\{\rho(x_n, x_k) : k > n\}, \qquad (n = 0, 1, \ldots),$$

(3) $$\delta_n \leqq \frac{1}{1-q} \cdot \rho(x_n, x_{n+1}), \qquad (n = 0, 1, \ldots).$$

Proof. Since for $n = 0, 1, \ldots$

(4) $$\delta_n = \max[\sup\{\rho(x_n, x_k) : k > n\};\ \delta_{n+1}]$$

from (1) and (4) we have (2). For $m > n$ we have

(5) $$\rho(x_n, x_m) \leqq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_m) \leqq \rho(x_n, x_{n+1}) + \\ + \delta_{n+1} \leqq \rho(x_n, x_{n+1}) + \alpha \cdot \delta_n,$$

and from (2) and (5) we have (3). □

Remark 1. *It is easy to werify that each Picard iteration of a generalized quasi-contraction satisfies condition* (1).

For $x, y \in X$ we put

$$\triangle(x, y) = \operatorname{diam}\{x, y, f(x), f(y), f^2(x), f^2(y), \ldots\}.$$

Lemma 2. *If $f: X \to X$ is a generalised quasi-contraciton (with respect to α) then*

$$\rho(f^k(x), f^k(y)) \leqq \alpha^k \cdot \triangle(x, y)$$

holds for each positive integer k and all $x, y \in X$.

Proof. For $k > 1$ and $x, y \in X$ we have

$$\rho(f^k(x), f^k(y)) = \rho(f(f^{k-1}(x)), f(f^{k-1}(y))) \leqq \alpha \cdot \triangle(f^{k-1}(x), f^{k-1}(y)) \leqq \\ \leqq \alpha^2 \cdot \triangle(f^{k-2}(x), f^{k-2}(y)) \leqq \cdots \leqq \alpha^k \cdot \triangle(x, y). \quad \square$$

Let $f: X \to X$ be a generalized quasi-contraciton.

Remark 2. *From Lemma 2 we have for all $x, y \in X$*

(6) $$\triangle(x, y) = \max\{\rho(x, y);\ \rho^*(x, f(x));\ \rho^*(y, f(y));\ \rho^*(x, f(y));\ \rho^*(y, f(x))\}$$

and from (6) *we have that the conditon* (C*) *is equivalent to* (C) *and by Remark* 1 *from* (2) *we have*

(7) $$\triangle(x, y) \leqq \rho(x, y) + \max\{\triangle(x, x);\ \triangle(y, y)\}.$$

2. Sequences of generalized quasi-contractions

Theorem 1. *Let $f_n: X \to X$ be a mapping with at least one fixed poit x_n for each $n = 1, 2, \ldots$ and $f: X \to X$ be a generalized quasi-contraction with fixed point x^*. If the sequence $\{f_n\}$ converges uniformly to f. then the suquence $\{x_n\}$ converges to x^*.*

Proof. Choose an arbitrary positive integer ε. Corresponding to this choice we can find a positive integer N such that

$$\rho(f_n(x_n), f(x_n)) < \varepsilon \cdot (1-\alpha)^2 \qquad (n > N), \tag{8}$$

since $\{f_n\}$ converges uniformly to f. For $n > N$ we have

$$\rho(x_n, x^*) = \rho(f_n(x_n), f(x^*)) \leqq \rho(f_n(x_n), f(x_n)) + \tag{9}$$
$$+\rho(f(x_n), f(x^*)) \leqq \rho(f_n(x_n), f(x_n)) + \alpha \cdot \triangle(x_n, x^*).$$

From (9) and (7) we have

$$\rho(x_n, x^*) \leqq \rho(f_n(x_n), f(x_n)) + \alpha \cdot (\rho(x_n, x^*) + \triangle(x_n, x_n)). \tag{10}$$

From (10) and (3) by Remark 1 we have

$$\rho(x_n, x^*) \leqq \rho(f_r(x_n), f(x_n)) + \alpha \cdot \left(\rho(x_n, x^*) + \frac{1}{1-\alpha} \cdot \rho(x_n, f(x_n))\right), \tag{11}$$

and we have

$$\rho(x_n, x^*) \leqq \frac{1}{(1-\alpha)^2} \cdot \rho(f_n(x_n), f(x_n)), \tag{12}$$

and from (12) and (8) we have $\rho(x_n, x^*) < \varepsilon$. □

3. Fixed point interations using infinite matrices and generalized quasi-contractions.

Let X be a normed linear space, f a mapping of X into itself and $A = (a_{,k})$ be an infinite matrix. Define the iteration scheme

$$\bar{x}_0 = x_0 \in X, \tag{13}$$

$$\bar{x}_{n+1} = f(x_n) \qquad (n = 0, 1, \ldots), \tag{14}$$

$$x_n = \sum_{k=0}^{n} a_{nk} \bar{x}_k \qquad (n = 1, 2, \ldots). \tag{15}$$

The scheme (13—15) is generally known as *Mann* [10] process.

Recently *J. Reinermann* [13] has defined summability matrix A by

$$a_{nk} = \begin{cases} c_k \cdot \prod_{j=k+1}^{n} (1 - c_j) & (k < n), \\ c_r & (k = n), \\ 0 & (k > n), \end{cases} \tag{16}$$

where the real sequence $\{c_n\}$ satisfies the next conditions

$$c_0 = 1, \tag{17}$$

$$0 < c_n < 1, \tag{18}$$

$$\sum_{n=0}^{\infty} c_n = \infty. \tag{19}$$

He then defined the iteration scheme (16) which can be written in the form

$$x_{n+1}=(1-c_n)\cdot x_n+c_n\cdot f(x_n) \qquad (n=0,1,\ldots). \tag{20}$$

The same iteration scheme has been defined independently by *C. L. Outlaw and C. W. Groetsch* [11] and *W. G. Dotson* [6]. Recently *J. Achari* [1] proved a theorem for fixed point iterations (20) concerning *Ćirić's* quasi-contraction. In this paper we extend his results for generalized quasi-contractions.

Theorem 2. *Let M be a non-empty closed bounded convex subset of X, f a mapping of M into itself and satisfies* (C*). *Suppose that A is an infinite matrix defined by* (16) *with $\{c_n\}$ satisfying* (17), (18) *and bounded away from zero. If the sequence, defined by* (20) *converges to x^* then x^* is the unique fixed point of f in M.*

Proof. For each n

$$x_{n+1}-x_n=c_n\cdot(f(x_n)-x_n).$$

Since

$$\lim_{n\to\infty} x_n=x^*,$$

$$\lim_{n\to\infty} \|x_{n+1}-x_n\|=0,$$

then

$$\lim_{n\to\infty} \|f(x_n)-x_n\|=0, \tag{21}$$

because $\{c_n\}$ is bounded away from zero. By Remark 1 from (3) we have

$$\Delta(x_n, x_n)\leqq\frac{1}{1-\alpha}\|x_n-f(x_n)\| \tag{22}$$

and from (22) and (21) we have

$$\lim_{n\to\infty} \Delta(x_n, x_n)=0. \tag{23}$$

For arbitrary fixed positive integer k from (23) we get

$$\lim_{n\to\infty} \|x_n-f^k(x_n)\|=0, \tag{24}$$

and by Remark 1 from (7) we have

$$\|f^k(x_n)-f^k(x^*)\|\leqq\alpha^k\cdot\Delta(x_n, x^*)\leqq\alpha^k\|x^*-x_n\|+ \\ +\alpha^k\cdot\Delta(x_n, x_n)+\alpha^k\cdot\Delta(x^*, x^*), \tag{25}$$

and we have

$$\limsup_{n\to\infty} \|f^k(x_n)-f^k(x^*)\|\leqq\alpha\cdot\Delta(x^*, x^*). \tag{26}$$

Since

$$\|x^*-f^k(x^*)\|\leqq\|x^*-x_n\|+\|x_n-f^k(x_n)\|+\|f^k(x_n)-f^k(x^*)\| \tag{27}$$

by Remark 1 from (2), (24), (26) and (27) we have $\Delta(x^*, x^*)\leqq\alpha\cdot\Delta(x^*, x^*)$. Since $0\leqq\alpha<1$ we have $\Delta(x^*, x^*)=0$, thus $x^*=f(x^*)$. □

[1. Theorem 4] is a special case of our theorem.

REFERENCES

[1] J. Achari, *Some results on Ciric's quasi-contraction mappings.* Publ. l'Inst. Math. (Beograd) Nouvelle série, tome **21** (35) 1977. pp. 9—14.

[2] S. Banach, *Sur les operations dan les ensembles abstraits et leur aplications aux équations intégrales,* Fund. Math. 3 (1922), pp. 133—181.

[3] Lj. B. Ćirić, *Generalized contractions and fixed-point theorems,* Publ. l'Inst. Math. (Beograd), Nouvelle série, tome **12** (26), 1971. pp. 19—26.

[4] Lj. B. Ćirić, *A generalization of Banach's contraction principle,* Proc Amer. Math. Soc **45**, Number 2, (1974), pp. 267—273.

[5] S. K. Chatterjea, *Fixed-point theorems,* C. R. Acad. Bulgare Sci **25**. (1972), pp. 727—730.

[6] W. G. Dotson Jr. *On the Mann interative process.* Trans Amer. Math. Soc. **149** (1970), pp. 65—73.

[7] G. E. Hardy and T. D. Rogers, *A generalization of a fixed point theorem of Reich,* Canad Math. Bull. Voll. **16** (2), 1973, pp. 201—206.

[8] M. Hegedüs, *A new generalization of Banach's contraction principle,* (to appear in Acta Sci Math. (Szeged)).

[9] R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc. **60** (1968), pp. 71—76.

[10] W. R. Mann, *Mean value methods in interation,* Proc. Amer. Math. Soc. **4** (1953) pp. 506—510.

[11] C. L. Outlaw and C. W. Groetsch, *Averaging iteration in a Banach space,* Bull. Amer. Math. Soc. **75** (1969), pp. 430—432.

[12] S. Reich, *Fixed poiuts of contractive functions,* Boll. Un. Math. Ital. **4** (5), 1972, pp. 24—26.

[13] J. Reinermann, *Über Toeplitzsch Iterationverfahren und einige Anwendungen in der konstruktiven Fixpunkttheorie,* Studia Math. **32** (1969), pp. 209—227.

[14[B. E. Rhoades, *A comparison of various definitions of contractive mappings,* Trans. Amer. Math. Soc. **226** (1977), pp. 256—290.

[15] T. Zamfirescu, *A theorem on fixed points.* Atti. Acad. Naz. Lincei Rend. Cl. Sci. Fis. Math. Natur. (8), **52** (1972), pp. 832—834.

M. Hegedüs, Department of Mathematics, Karl Marx University for Economics Budapest, IX. Dimitrov tér 8, Hungary. 1093.