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GRAPHS WITH GREATEST NUMBER OF MATCHINGS

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In the present paper finite graphs without loops and multiple edges will be considered. If not stated otherwise, the vertices of a graph G will be labeled by $v_j = v_j(G)$, $j = 1, 2, \ldots$. The egde connecting the vertices v_r and v_s is denoted by e_{rs}.

If G and H are isomorphis, we shall write $G = H$. The direct sum (or union) of graphs G_1 and G_2 is denoted by $G_1 \dot{+} G_2$.

Notation and terminologv not introduced here follows the book [8].

The path and the cycle with n vertices will be denoted by P_n and C_n, respectively. P_1 is just an isolated vertex. The vertices of P_n and C_n will be labeled so that v_j and v_{j+1} are adjacent $(j = 1, 2, \ldots, n-1)$. Thus $P_n + e_{1n} = C_n$ $(n \geqslant 3)$.

Let G and H be two disjoint graphs. Then the graph $G(r, s) H$ is obtained by connecting the vertices $v_r(G)$ and $v_s(H)$ by a new edge. The graph $P_a(1, 1)\ C(s, 1) P_b$ is constructed by joining the vertices v_1 and v_s of the cycle C_n to the terminal vertices $v_1(P_a)$ and $v_1(P_b)$ of P_a and P_b, respectively. The graph $C_a(1, 1) P_n(n, 1) C_b$ is constructed by joining the terminal vertices v_1 and v_n of P_n to (arbitrary) vertices $v_1(C_a)$ and $v_1(C_b)$ of C_a and C_b, respectively (see Fig. 1).

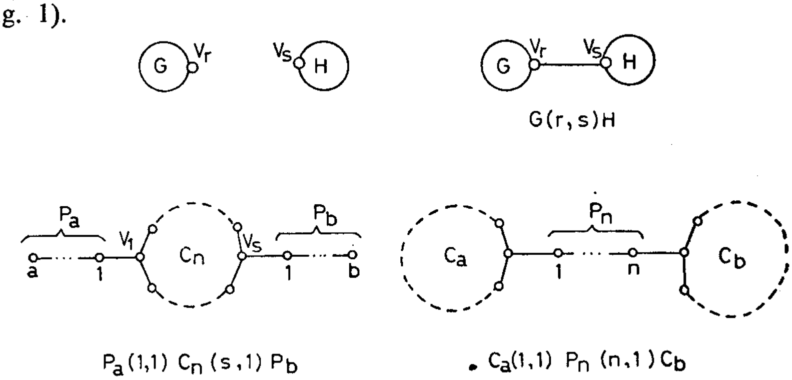

Fig. 1

Let v_r and v_s be two adjacent vertices of a graph G with n vertices. The substitution of the edge e_{rs} by a path with a vertices yields the graph $G(e_{rs} | a)$ with $n+a$ vertices.

The dot product $C_a \cdot C_b$ of the cycles C_a and C_b is obtained by identifying a vertex of C_a with a vertex of C_b.

Let P_a, P_b and P_c be three disjoint paths ($a \geqslant 3$, $b \geqslant 3$, $c \geqslant 3$). By identifying the vertices $v_1(P_a)$, $v_1(P_b)$ and $v_1(P_c)$ and by simultaneous identifying the vertices $v_a(P_a)$, $v_b(P_b)$ and $v_c(P_c)$ one obtains a bicyclic graph $Q(a, b, c)$ with $a+b+c-4$ vertices (see Fig. 2).

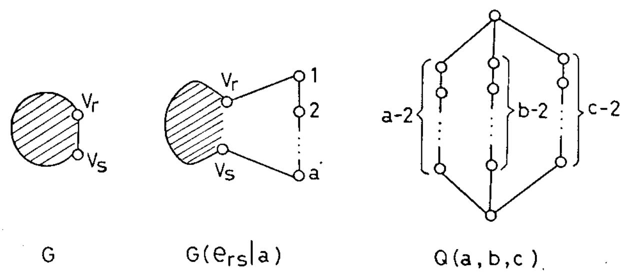

Fig. 2

Definition 1. A subgraph of G induced by k independet edges is called a k-matching of G. The number of k-matchings in G is denoted by $p(G, k)$.

It is both convenient and consequent to define $p(G, 0)=1$ for all graphs G.

The numbers $p(G, k)$ play an important role in various chemical [1, 5, 6, 7, 10] and physical [9] theories. They have been subject also to several mathematical investigations [2, 3, 4, 11]. We mention here only the following results.

1. For every graph G there exists a number $K=K(G)$, such that [11] $p(G, 1) \leqslant p(G, 2) \leqslant \cdots \leqslant p(G, K) \geqslant p(G, K+1) \geqslant p(G, K+2) \geqslant \cdots$.

2. The matching polynomial of a graph G,

$$\alpha(G) = \sum_{k=0}^{[n/2]} (-1)^k p(G, k) \lambda^{n-2k}$$

coincides with the characteristic polynomial of this graph if and only if G is a forest [4].

3. All the zeros of $\alpha(G)$ are real [3,9].

4. The recurrence relation [2,4]

$$p(G, k) = p(G - e_{rs}, k) + p(G - v_r - v_s, k-1) \tag{1}$$

will be frequently used later.

Since P_1 has no edges, $p(G \dot{+} P_1, k) = p(G, k)$.

We introduce now a quasi-ordering of graphs according to the number of matchings in them.

Definition 2. For two graphs G and H we write $G \succ H$ if $p(G, k) \geqslant p(H, k)$ for all $k=1, 2, \dots$. If $G \succ H$ and $H \succ G$, then we call the graphs G and H matching equivalent and write $G \sim H$.

Combining Definitions 1 and 2 one immediately arrives to the following two conclusions.

Lemma 1. *If H is a subgraph of G, then $G \succ H$. Moreover if the edge set of H is a proper subset of the edge set of G, then G and H are not matching equivalent.*

Lemma 2. $G \dot{+} P_1 \sim G$.

Let γ be a set of graphs. Then the relation $\dot{\sim}$ is an equivalence relation in this set. The corresponding equivalence classes will be called the matching equivalence classes (of the set γ). Clearly, the quasiordering $\succ$ induces a partial ordering in $\gamma/\sim$.

Let $\gamma_1, \gamma_2, \dots$ be the matching equivalence classes of γ. As usual, a class γ_i is called the greatest class if $\gamma_i \succ \gamma_j$ for all $j=1, 2, \dots$. This maximal class (provided it exists) will be denoted by γ_1. The graphs from γ_1 will be said to have greatest number of matchings in the set γ. When ambiguities are avoided, the elements of γ_1 will be simply called the greatest graphs in γ.

γ_2 is the second greatest class in γ (and its elements are the second greatest graphs in γ) if $\gamma_2 \succ \gamma_j$ for all $j=2, 3, \dots$.

The third greatest, fourth greatest etc. matching equivalence classes and the third greatest, fourth greatest etc. graphs are determined analogously.

Of course, greatest classes in a set of graphs need not exist at all. The ›mallest pair of graphs which are mutually incomparable with respect to the ɜlation $\succ$ are C_3 and $P_2 \dot{+} P_2$.

We denote by $\gamma(n)$ the set of all graphs with n vertices and by $\beta(a, b)$ ıe set of all bipartite graphs with $a+b$ vertices. Further, $\Gamma(n)=\bigcup_{j=1}^{n}\gamma(j)$ and $(a, b)=\bigcup_{i=1}^{a}\bigcup_{j=1}^{b}\beta(i, j)$. The set of all graphs with n vertices and cyclomatic ımber c is $\gamma(n, c)$ and $\Gamma(n, c)=\bigcup_{j=1}^{n}\gamma(j, c)$.

In a previous paper [5] it was established that the unique greatest and e unique second greatest graphs in the set $\gamma(n, 0)$ are P_n and $P_{n-2}(3, 1)P_2$, spectively. This result can be slighty improved as follows.

Theorem 1. *P_n and $P_{n-2}(3, 1)P_2$ are the unique greatest and the unique second greatest graphs, respectively, in the set $\Gamma(n, 0)$.*

We present here without proof also the following two results.

Theorem 2. (a) *If $n \geqslant 1$, the complete graph K_n with n vertices has the greatest number of matchings in $\Gamma(n)$.* (b) *If $n \geqslant 2$, $K_n - e$ is the unique second greatest graph in the same set.* (c) *If $n \geqslant 4$, $K_n - e_1 - e_2$ is the unique third greatest graph in the same set with e_1 and e_2 being non-incident edges of K_n.* (d) *If $n=3$, the third greatest matching equivalence class in $\Gamma(3)$ is $\{P_2 \dot{+} P_1, P_2\}$.*

Theorem 3. (a) *If* $a \geqslant 1$ *and* $b \geqslant 1$, *the complete bipartite graph* $K_{a,b}$ *has the greatest number of matchings in* $\mathrm{B}(a, b)$. (b) *If* $a \geqslant 2$ *and* $b \geqslant 2$ *then* $K_{a,b} - e$ *is the unique second greatest graph in the same set.* (c) *If* $a \geqslant 2$, $b \geqslant 2$, $K_{a,b} - e_1 - e_2$ *is the unique third greatest graph in the same set, where* e_1 *and* e_2 *are non-incident edges.* (d) *If* $a \geqslant 2$ *and* $b = 1$, *the second and third greatest matching equivalence classes are* $\{K_{a,1} - e,\ K_{a-1,1}\}$ *and* $\{K_{a,1} - e_1 - e_2,\ K_{a-1,1} - e,\ K_{a-2,1}\}$, *respectively.*

We proceed now to determine the unicyclic and bicyclic graphs with greatest number of matchings. For this purpose we shall formulate three auxiliay results.

Lemma 3. $G(r, s) H \succ G \dot{+} H$.

The above statement is a special case of Lemma 1. Its consequence is that for every graph $G \in \gamma(n, c)$ there exists a connected graph $G_1 \in \gamma(n, c)$, such that $G_1 \succ G$.

Lemma 4. *Let* F *be a forest with* a *vertices and* G *an arbitrary graph. Let* v_r *and* v_s *be vertices of* G *and* F, *respectively. Then* $G(r, 1) P_a \succ G(r, s) F$.

Proof. Applying eq. (1) to the edge e_{rs} of $G(r, s) F$ one gets

$$p(G(r, s) F, k) = p(G \dot{+} F, k) + p((G - v_r) \dot{+} (F - v_s), k - 1).$$

Similary,

$$p(G(r, 1) P_a, k) = p(G \dot{+} P_a, k) + p((G - v_r) \dot{+} P_{a-1}, k - 1).$$

From Theorem 1, $G \dot{+} P_a \succ G \dot{+} F$ and $(G - v_r) \dot{+} P_{a-1} \succ (G - v_r) \dot{+} (F - v_s)$ and Lemma 4 follows.

Lemma 5. *Let* G *be an arbitrary graph and let* v_r *and* v_s *be its two adjacent vertices. Then* $G(e_{rs} | a) \succ G(r, 1) P_a$.

Proof. Let us for brevity denote $G(e_{rs} | a)$ by H. Note that

$$H - e_{1e} - e_{as} = (G - e_{rs}) \dot{+} P_a; \quad H - e_{1r} - v_a - v_s = (G - v_s) \dot{+} P_{a-1};$$

$$H - v_1 - v_r - e_{as} = (G - v_r) \dot{+} P_{a-1}$$

and

$$H - v_1 - v_r - v_a - v_s = (G - v_r - v_s) \dot{+} P_{a-2}.$$

Then a repeated application of eq. (1) gives

$$p(G(r, 1) P_a, k) = p((G - e_{rs}) \dot{+} P_a, k) + p((G - v_r - v_s) \dot{+} P_a, k - 1) + \\ + p((G - v_r) \dot{+} P_{a-1}, k - 1)$$

and

$$p(H, k) = p((G - e_{rs}) \dot{+} P_a, k) + p((G - v_s) \dot{+} P_{a-1}, k - 1) + \\ + p((G - v_r) \dot{+} P_{a-1}, k - 1) + p((G - v_r - v_s) \dot{+} P_{a-2}, k - 2).$$

Now, since $G - v_r - v_s$ is a subgraph of $G - v_s$, we have further

$$p((G - v_s) \dot{+} P_{a-1}, k - 1) + p((G - v_r - v_s) \dot{+} P_{a-2}, k - 2) \geqslant \\ \geqslant p(G - v_r - v_s) \dot{+} P_{a-1}, k - 1) + p((G - v_r - v_s) \dot{+} P_{a-2}, k - 2) = \\ = p((G - v_r - v_s) \dot{+} P_a, k - 1).$$

Therefore, $p(G(r, 1) P_a, k) \leqslant p(G(e_{rs} | a), k)$.

A consequence of Lemmas 4 and 5 is that for every graph $G\in\gamma(n, c)$, $c>0$, there exists a graph $G_1\in\gamma(n, c)$ without vertices of degree one, such that $G_1\succ G$. Accordingly, from Lemmas $3-5$ it follows that the greatest matching equivalence class of $\gamma(n, c)$, $c>0$, possesses elements which are connected graphs without vertices of degree one.

Theorem 4. (a) *If* $n\geqslant 3$, *the cycle* C_n *has the greatest number of matchings in the set* $\gamma(n, 1)$. (b) *If* $n\geqslant 5$, *the second greatest matching equivalence class of the same set is* $\{C_{n-2}(1, 1)P_2, C_4(1, 1)P_{n-4}\}$. (c) *If* $n\geqslant 7$, *the third greatest matching equivalence class of the same set is* $\{C_{n-4}(1,1)P_4, C_6(1,1)P_{n-6}\}$.

Remark 1. The matching equivalence classes under (b) and (c) contain a single graph for $n=6$ and $n=10$, respectively.

Remark 2. If $n=4$, the second and third greatest classes are $\{C_3(1,1)P_1\}$ and $\{C_3\dot{+}P_1, C_3\}$, respectively. If $n=5$ and $n=6$, then the third greatest classes are $\{P_5+e_{24}\}$ and $\{C_3(1,1)P_3, C_5(1,1)P_1\}$, respectively.

First we prove two preliminary results.

Lemma 6.

$$C_n\succ C_{n-2}(1,1)P_2\sim C_4(1,1)P_{n-4}\succ C_{n-4}(1,1)P_4\sim C_6(1,1)P_{n-6}\succ C_{n-j}(1,1)P_j$$

for all other values of j.

Proof. Provided that $0\leqslant j\leqslant n-3$, one deduces from (1)

$$p(C_{n-j}(1,1)P_j, k)=p(P_n, k)+p(P_j\dot{+}P_{n-j-2}, k-1).$$

It is proved in [5] that

$$P_n\succ P_2\dot{+}P_{n-2}\succ P_4\dot{+}P_{n-4}\succ P_j\dot{+}P_{n-j}$$

for

$$j=1,\ 3,\ 5,\ 6,\ 7,\ \ldots.$$

Therefore $p(P_j\dot{+}P_{n-j-2}, k-1)$ reaches its maximal, second maximal and third maximal value for $j=0$, $j=2$ or $n-4$ and $j=4$ or $n-6$, respectively.

Lemma 7. *Let* $1\leqslant s\leqslant n$. *Then*

$$C_n(1,1)P_{a+b}\succ P_a(1,1)C_n(2,1)P_b\succ P_a(1,1)C_n(s, 1)P_b$$

for all values of n, a, b.

Proof. Application of (1) gives

$$p(C_n(1,1)P_{a+b}, k)=p(P_b\dot{+}C_n(1,1)P_a, k)+$$
$$+p(P_{b-1}\dot{+}P_{n-a-1}, k-1)+p(P_{b-1}\dot{+}P_{a-1}\dot{+}P_{n-2}, k-2)$$

and

$$p(P_a(1,1)C_n(2,1)P_b, k)=p(P_b\dot{+}C_n(1,1)P_a, k)+$$
$$+p(P_{b-1}\dot{+}P_{n-a-1}, k-1)$$

from which the left relation is evident.

Let $s \neq 1$. Then according to our labeling, in C_n there are $s-2$ vertices between v_1 and v_s. Then

$$p(P_a(1,1)\,C_n(s,\,1)\,P_b,\,k)=p(P_b\dot{+}C_n(1,1)\,P_a,\,k)+$$

$$+p(P_a\dot{+}P_{b-1}\dot{+}P_{n-1},\,k-1)+p(P_{a-1}\dot{+}P_{b-1}\dot{+}P_{s-2}\dot{+}P_{n-s},\,k-2).$$

The number $p(P_a(1,1)\,C_n(s,\,1)\,P_b,\,k)$ is maximal if

$$p(P_{a-1}\dot{+}P_{b-1}\dot{+}P_{s-2}\dot{+}P_{n-s},\,k-2)$$

is maximal. From Theorem 1 we see that this will occur when $s=2$ or $s=n$, i. e. when the vertices v_1 and v_s are adjacent. This proves the right relation of Lemma 7 for $s\neq 1$. From

$$p(P_a(1,1)\,C_n(2,1)\,P_b,\,k)=p(P_b\dot{+}C_n(1,1)\,P_a,\,k)+p(P_{b-1}\dot{+}P_{n+a-1},\,k-1)$$

and

$$p(P_a(1,1)\,C_n(1,1)\,P_b,\,k)=p(P_b\dot{+}C_n(1,1)\,P_a,\,k)+p(P_{b-1}\dot{+}P_{n-1}\dot{+}P_a,\,k-1)$$

follows the validity of the same relation for $s=1$.

P r o o f o f T h e o r e m 4. In every unicyclic graph G with n vertices there exists an edge e_{rs} such $G-e_{rs}$ and $G-v_r-v_s$ are forests with n and $n-2$ vertices, respectively. By (1) and Theorem 1, the graph G will be greatest if $G-e_{rs}=P_n$ and $G-v_r-v_s=P_{n-2}$. It is easily seen that the above identities are fulfilled only in the case of the cycle C_n. This proves statement (a).

From Lemmas $3-5$ it follows that the greatest unicyclic graph with n vertices (wich is not C_n) must be of the form $C_{n-j}(1,1)\,P_j$ $(j\neq 0)$. Lemma 6 guarantees that the greatest graphs in this class are $C_4(1,1)\,P_{n-4}$ and $C_{n-2}(1,1)\,P_2$. Statement (b) follows.

According to Lemmas $3-5$, the third greatest unicyclic graphs with n vertices must be ot the type

$$C_{n-a}(1,1)\,P_a\ \ (a\neq 0,\ a\neq 2,\ n-a\neq 4)$$

or

$$P_a(1,1)\,C_{n-a-b}(s,\,1)\,P_b.$$

Lemma 6 proves that among the graphs $C_{n-a}(1,1)\,P_a$ only the pair $C_{n-4}(1,1)\,P_4 \sim C_6(1,1)\,P_{n-6}$ is to be considered as a canditate for third greatest unicyclic graphs. From Lemmas 6 and 7 we know that $C_{n-4}(1,1)\,P_4 \succ$ $\succ P_a(1,1)\,C_{n-a-b}(s,\,1)\,P_b$ whenever $a+b>2$. Lemma 7 also shows that a graph of the type $P_a(1,1)\,C_{n-a-b}(s,\,1)\,P_b$ can be greatest only if $s=2$ or (what is the same) $s=n$.

Let $\gamma_3(n,\,1)$ denote the third greatest matching equivalence class of $\gamma(n,\,1)$. If $\gamma_3(n,\,1)$ exists, then according to the above consideration it must be a subset of

$$\{C_{n-4}(1,1)\,P_4,\ C_6(1,1)\,P_{n-6},\ P_1(1,1)\,C_{n-2}(2,1)\,P_1\}.$$

It is now easy to show that $C_6(1,1)\,P_{n-6}$ is, but $P_1(1,1)\,C_{n-2}(2,1)\,P_1$ is not matching equivalent with $C_{n-4}(1,1)\,P_4$. Moreover it is

$$C_{n-4}(1,1)\,P_4 \succ P_1(1,1)\,C_{n-2}(2,1)\,P_1.$$

Namely,

$$p(C_{n-4}(1,1)P_4,\ k)=p(P_n,\ k)+p(P_4\dot{+}P_{n-7},\ k-1)+p(P_4\dot{+}P_{n-8},\ k-2)$$

while

$$p(P_1(1,1)C_{n-2}(2,1)P_1,\ k)=p(P_n,\ k)+p(P_3\dot{+}P_{n-7},\ k-1)+p(P_2\dot{+}P_{n-8},\ k-2).$$

Statement (c) is proved.

Theorem 4 is thus proved. In addition we note that C_n has greatest number of matchings in $\Gamma(n,\ 1)$. This result follows from theorem 4a and the fact that by Lemma 2 every graph $G\in\gamma(m,\ 1)$, $m<n$ is matching equivalent with the graph $G\dot{+}P_1\dot{+}\cdots\dot{+}P_1\in\gamma(n,\ 1)$.

Theorem 5. *If $n\leqslant 9$, the unique graphs with the greatest number of matchings in the set $\Gamma(n,\ 2)$ are those presented in Fig. 3. If $n\geqslant 10$, there exists no greatest matching equivalence class in $\Gamma(n,\ 2)$, but two maximal ones:*

$$\Gamma_a(n,\ 2)=\{Q(4,\ 2,\ n-2)\}$$

and

$$\Gamma_b(n,\ 2)=\{C_4(1,1)P_{n-8}(n-8,1)C_4\}.$$

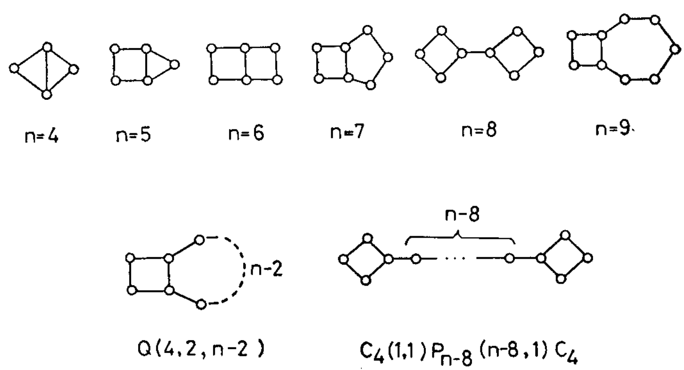

Fig. 3

Before proving Theorem 5 we shall formulate the Lemmas 8—10.

Lemma 8. $C_4\cdot C_{n-3}\succ C_j\cdot C_{n-j+1}$ *for all values of* $j=3,\ 5,\ 6,\ \ldots$.

Proof will be performed by induction on the total number n of vertices of $C_j\cdot C_{n-j+1}$. For $n=5$ and 6 there exists only one graph of this type and lemma is trivially true. Lemma 8 can be also easily verified for $n=7$. Supose then that the lemma holds for graphs $C_j\cdot C_{m-j+1}$ for all $m=7,\ 8,\ \ldots,\ n-1$.

Now,

$$p(C_j\cdot C_{n-j+1},\ k)=p(P_1(1,1)C_j(1,1)P_{n-j-1},\ k)+$$
$$+p(C_j(1,1)P_{n-j-2},\ k-1)=p(P_1(1,1)C_j(1,1)P_{n-j-2},\ k)+$$
$$+p(P_1(1,1)C_j(1,1)P_{n-j-3},\ k-1)+p(C_j(1,1)P_{n-j-3},\ k-1)+$$
$$+p(C_j(1,1)P_{n-j-4},\ k-2)=p(C_j\cdot C_{n-j},\ k)+p(C_j\cdot C_{n-j-1},\ k-1).$$

According to the induction hypothesis, $p(C_j \cdot C_{n-j}, k)$ and $p(C_j \cdot C_{n-j-1}, k-1)$ are maximal for $j=4$. Then also $p(C_j \cdot C_{n-j+1}, k)$ will be maximal for $j=4$.

Lemma 9. *$Q(4, 2, n-2) \succ Q(a, b, c)$ for all values of a, b and c, provided that $a+b+c-4=n$.*

Proof. Let e_{12} be the edge between the vertices $v_1(P_c)$ and $v_2(P_c)$ of the graph $Q(a, b, c)$. Then

$$Q(a, b, c) - e_{12} = C_{a+b-2}(1,1)P_{c-2}$$

and

$$Q(a, b, c) - v_1(P_c) - v_2(P_s) = P_{a+c-4}(a-1,1)P_{b-2}.$$

Consequently,

$$p(Q(a, b, c), k) = p(C_{a+b-2}(1,1)P_{c-2}, k) + p(P_{a+c-4}(a-1,1)P_{b-2}, k-1).$$

Theorem 1 implies that $p(P_{a+c-4}(a-1,1)P_{b-2}, k-1)$ is maximal if $b-2=0$. According to Lemma 6, $p(C_{a+b-2}(1,1)P_{c-2}, k)$ is maximal if $a+b-2=4$. Therefrom we conclude that $Q(a, b, c)$ is maximal for $a=4$, $b=2$ and $c=n-2$.

Lemma 10.

$$C_4(1,1)P_{n-8}(n-8,1)C_4 \succ C_a(1,1)P_{n-a-b}(n-a-b, 1)C_b$$

for all values of a, b and n.

This results can be deduced from similar arguments as used in the proof of Lemma 6.

Proof of Theorem 5. Because of Lemma 2 it is sufficient to consider graphs from $\gamma(n, 2)$.

From the Lemmas 3–5 follows that if greatest graphs exist in $\gamma(n, 2)$, then they must be of the form $Q(a, b, c)$ $(a+b+c=n-4)$ or $C_a(1,1)P_c(c, 1)C_b$ $(a+b+c=n)$ or $C_a \cdot C_b$ $(a+b=n+1)$. Lemmas 8–10 reduce the candidates for greatest graphs to the following tree: $Q(4, 2, n-2)$, $C_4(1,1)P_{n-8}(n-8, 1)C_4$ and $C_4 \cdot C_{n-3}$. For $n=4$, 5, 6 and 7, Theorem 5 can be verified by direct calculation.

We show now that

$$Q(4, 2, n-2) \succ C_4 \cdot C_{n-3}$$

and

$$C_4(1,1)P_{n-8}(n-8,1)C_4 \succ C_4 \cdot C_{n-3}$$

for all $n \geq 8$. This follows immediately from the fact that

$$p(Q(4, 2, n-2), k) = p(P_n, k) + 2\,p(P_2 \dot{+} P_{n-4}, k-1) + p(P_{n-5}, k-2)$$

$$p(C_4(1,1)P_{n-8}(n-8,1)C_4, k) = p(P_n, k) + 2\,p(P_2 \dot{+} P_{n-4}, k-1) + p(P_2 \dot{+} P_2 \dot{+} P_{n-8}, k-2)$$

and

$$p(C_4 \cdot C_{n-3}, k) = p(P_n, k) + p(P_2 \dot{+} P_{n-4}, k-1) + p(P_3 \dot{+} P_{n-5}, k-1).$$

Consequently, in the set $\Gamma(n, 2)\setminus\{Q(4, 2, n-2), C_4(1,1)P_{n-8}(n-8,1)C_4\}$ there cannot exist graphs G such that $G \succ Q(4, 2, n-2)$ and/or $G \succ C_4(1,1)P_{n-8}(n-8,1)C_4$. Therefore, the greatest graphs in $\Gamma(n, 2)$ are either $Q(4, 2, n-2)$ or $C_4(1,1)P_{n-8}(n-8, 1)$ or both, provided that they are comparable (with respect to the relation $\succ$). If the above two graphs are not comparable, then they will belong to maximal, but not greatest matching equivalence classes $\Gamma_a(n, 2)$ and $\Gamma_b(n, 2)$.

We see from the previous expressions that $Q(4, 2, n-2)$ and $C_4(1,1)P_{n-8}$ $(n-8, 1)C_4$ are comparable if and only if P_{n-5} and $P_2 \dot{+} P_2 \dot{+} P_{n-8}$ are comparable.

The graphs P_{n-5} and $P_2 \dot{+} P_2 \dot{+} P_{n-8}$ are comparable for $n=8$ and $n=9$, viz. $P_2 \dot{+} P_2 \succ P_3$ and $P_4 \succ P_2 \dot{+} P_2 \dot{+} P_1$. Accordingly, the greatest bicyclic graphs with eight and nine vertices are $C_4(1,1)C_4$ and $Q(4, 2, 7)$, respectively.

For $n \geqslant 10$ the graphs P_{n-5} and $P_2 \dot{+} P_2 \dot{+} P_{n-8}$ are mutually incomparable. This can be seen from

$$p(P_{n-5}, 1) = n-6 > p(P_2 \dot{+} P_2 \dot{+} P_{n-8}, 1) = n-7$$

and

$$p(P_{n-5}, (n-4)/2) = 0 < p(P_2 \dot{+} P_2 \dot{+} P_{n-8}, (n-4)/2) = 1$$

if n is even, or

$$p(P_{n-5}, (n-5)/2) = 1 < p(P_2 \dot{+} P_2 \dot{+} P_{n-8}, (n-5)/2) = (n-7)/2$$

if n is odd.

The uniqueness of the element in the classes $\Gamma_a(n, 2)$ and $\Gamma_b(n, 2)$ is guaranteed by the Lemmas 3—5 and 8—10.

* * *

As we have seen, the graphs which are greatest or maximal with respect to the relation $\succ$ are in a certain sense „unexpected". The search for maximal graphs in the sets $\gamma(n, c)$ for $c > 2$ will be a rather difficult task.

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