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# CHARACTERISTIC AND MATCHING POLYNOMIALS OF SOME COMPOUND GRAPHS

*Ivan Gutman*

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In the present work the characteristic and matching polynomials will be determined for a class of compound graphs. We shall adopt the same notation and terminology as in our previous paper [3]. The only difference is that the formerly proposed name "acyclic polynomial" will be changed into "matching polynomial".

The characteristic and the matching polynomial of a graph $G$ are denoted by $\Phi(G)=\Phi(G, \lambda)$ and $\alpha(G)=\alpha(G, \lambda)$, respectively. If a polynomial $f(\lambda)$ is a divisor of the characteristic polynomial of at least one graph, we say that $f(\lambda)$ is $\Phi$-graphic. If a polynomial $g(\lambda)$ is a divisor of the matching polynomial of at least one graph, we say that $g(\lambda)$ is $\alpha$-graphic.

The following properties of the characteristic and matching polynomials have been proved elsewhere [1, 2, 3, 4].

**1.** $\Phi(G)=\alpha(G)$ if and only if the graph $G$ is a forest [3].

Let $e_{rs}$ be an edge which is incident to the vertices $v_r$ and $v_s$.

**2.** If $e_{rs}$ is a bridge in $G$, then [2]

$$\Phi(G)=\Phi(G-e_{rs})-\Phi(G-v_r-v_s). \tag{1}$$

**3.** If $e_{rs}$ is any edge in $G$, then [3]

$$\alpha(G)=\alpha(G-e_{rs})-\alpha(G-v_r-v_s). \tag{2}$$

**4.** All the zeros of $\Phi(G)$ and $\alpha(G)$ are real [1, 4].

As a consequence of this latter property, if a polynomial is either $\Phi$-graphic or $\alpha$-graphic, then all its zeros are real.

**5.** If a graph $G$ is composed of $p_j$ disconnected parts $G_j$ ($j=1, 2, \ldots c$), we shall write $G=\bigoplus_{j=1}^{c} p_j G_j$. Then [1, 3]

(3)
$$\Phi(G)=\prod_{j=1}^{c} \Phi(G_j)^{p_j},$$

(4)
$$\alpha(G)=\prod_{j=1}^{c} \alpha(G_j)^{p_j}.$$

Let $R$ be a rooted graph, i.e. a graph whose particular vertex is labeled by $\nu_0$. Let $d=(d_1, d_2, \ldots, d_m)$ be an ordered $m$-tuple of positive integers. $K_1$ will denote the graph consisting of a single vertex. Then we define a graph $G_m=G_m(R, d)$ recursively in the following manner.

Definition. (a) $G_0=R$

(b) For $k=0, 1, \ldots, m-1$, the graph $G_{k+1}$ is obtained by connecting $K_1$ to the vertices $\nu_k$ of $d_{k+1}$ copies of the graphs $G_k$. The vertex of $K_1$ is labeled $\nu_k$ (see Fig. 1).

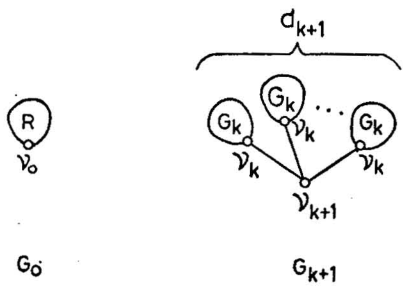


Fig. 1.

Hence, if $G_k$ possesses $n_k$ vertices, then $G_{k+1}$ possesses $n_{k+1}=d_{k+1}n_k+1$ vertices. According to the above definition, the graph $G_{k+1}$ can be presented as in Fig. 2.

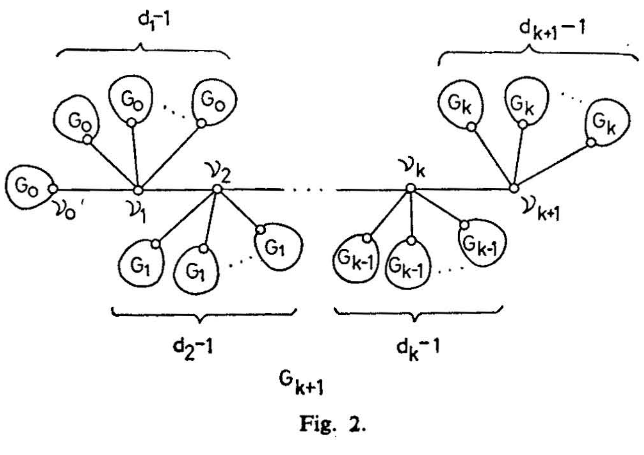


Fig. 2.

Note that in our notation the label $v_k$ $(k=0, 1, \dots, m-1)$ is not associated with a unique vertex, but with a class of mutually equivalent vertices. Nevertheless, this causes no ambiguity in the following considerations.

Heilmann and Lieb [4] have determined the matching polynomial of the graphs $G_m(R, d)$ in the special case when $R=K_1$ and $d=(d, d, \dots, d)$. These graphs are, of course, trees and therefore their matching and characteristic polynomials coincide. It is also worth noting that for $R=K_1$ and $d=(m, m-1, \dots, 1)$, the graph $G_m(R, d)$ coincides with a tree whose properties were recently investigated by Đ. Kurepa [5].

In the present paper the results of Heilmann and Lieb will be generalized and extended to the graphs $G_m(R, d)$ with arbitrary $R$ and $d$. Before starting with the calculation of $\Phi(G_m)$ and $\alpha(G_m)$ we point at the following property of the graphs $G_m(K_1, d)$, which can be deduced by elementary combinatorial arguments.

Propositon 1. In the graph $G_m(K_1, d)$ there exists a spanning subgraph $S$ of the from $S=\bigoplus_{k=1}^{m+1} a_k P_k$, where $P_k$ denotes a path with $k$ vertices and

$$a_k=\begin{cases}(d_k-1)\prod_{j=k+1}^{m} d_j & \text{for } 1\leqslant k\leqslant m-1,\\ d_m-1 & \text{for } k=m,\\ 1 & \text{for } k=m+1.\end{cases}$$

In other words, the vertices of $G_m(K_1, d)$ can be covered with a collection of path graphs, which contains $a_k$ copies of $P_k$ $(k=1, 2, \dots, m+1)$.

## *The characteristic and matching polynomials fo the graphs $G_m(R, d)$,*

The edge betwen the vertex $v_m$ and each of the $d_m$ vertices labeled by $v_{m-1}$ in $G_m$ is a bridge (see Fig. 1). Therefore from eqs. (1) — (4) it follows

$$\Phi(G_m)=\lambda[\Phi(G_{m-1})]^{d_m}-d_m[\Phi(G_{m-1})]^{d_m-1}\Phi(G_{m-1}-v_{m-1}) \tag{5}$$

and

$$\alpha(G_m)=\lambda[\alpha(G_{m-1})]^{d_m}-d_m[\alpha(G_{m-1})]^{d_m-1}\alpha(G_{m-1}-v_{m-1}). \tag{6}$$

Consequently, $[\Phi(G_{m-1})]^{d_m-1}$ is a factor of $\Phi(G_m)$ while $[\alpha(G_{m-1})]^{d_m-1}$ is a factor of $\alpha(G_m)$. In addition, from the Definition and eqs. (3) and (4) results that $\Phi(G_{m-1}-v_{m-1})=[\Phi(G_{m-2})]^{d_{m-1}}$ and $\alpha(G_{m-1}-v_{m-1})=[\alpha(G_{m-2})]^{d_{m-1}}$.

Since eqs. (5) and (6) are fully equivalent, we shall restrict our considerations only to the characteristic polynomial. Completely analogous results hold, however, also for the matching polynomials.

Now, from (5) is seen that $[\Phi(G_k)]^{d_{k+1}-1}$ is a factor of $\Phi(G_{k+1})$ for all $k=0, 1, \dots, m-1$. This can be written as

$$\Phi(G_k)=f(G_k)\prod_{j=1}^{k}[\Phi(G_{j-1})]^{d_j-1}, \tag{7}$$

where $f(G_k)=f(G_k, \lambda)$ is some polynomial ($k=0, 1, \dots, m$). Hence, $f(G_k)$ is a factor of $\Phi(G_m)$ for all $k=0, 1, \dots, m$. Straightforward calculation yields then the following result.

Proposition 2a. $\Phi(G_m)$ can be completely factorized in terms of the polynomials $f(G_k)$, $k=0, 1, \dots, m$, viz.

$$\Phi(G_m)=\prod_{k=0}^{m}[f(G_k)]^{a_{k+1}},$$

where the quantities $a_{k+1}$ are exactly the same as those defined Proposition 1

Hence, the determination of the characteristic polynomial of $G_m$ is fully reduced to the finding of the polynomials $f(G_k)$ for $k=0, 1, \dots, m$. Substitution of (7) back into (5) results immediately in a recurrence relation for $f(G_k)$.

Proposition 2b.

$$f(G_k)=f(G_{k-1})-d_k f(G_{k-2}) \tag{8}$$

with the initial conditions

$$f(G_0)=\Phi(G), \tag{9'}$$

$$f(G_1)=\lambda\,\Phi(G)-d_1\,\Phi(G-\nu_0). \tag{9''}$$

*Special cases*

There is no restriction on the numberss $d_k$ except that they must be positive integers. Therefore Proposition 2*b* provides a variety of possibilities for the construction of graphs $G_m$ with interesting characteristic and matching polynomials We would like to emphasize first the following two special cases.

If $d_k=k$ ($k=1, 2, \dots, m$), then eq. (8) closely resembles the recursion relation for the Hermite polynomials. If in addition $R=K_1$, then $f(G_0)=\lambda=H_1^*$ and $f(G_1)=\lambda^2-1=H_2^*$, where $H_k^*=H_k^*(\lambda)$ is the $k$-th Hermite polynomial. (We follow the notation of the book [6], p. 60.)

Proposition 3. If $d=(1, 2, \dots, m)$ and $R=K_1$, then the characteristic (and matching) polynomial of $G_m$ is completely factorized into Hermite polynomials,

$$\Phi(G)=\alpha(G_m)=\prod_{k=1}^{m+1}(H_k^*)^{a_k},$$

where

$$a_k=\begin{cases} m!\,(k-1)/k & \text{for } 1\leqslant k\leqslant m, \\ 1 & \text{for } k=m+1. \end{cases}$$

Proposition 4. If $d=(1, 1, 2, 2, 3, \ldots)$, i.e. $d_k=\left[\frac{k+1}{1}\right]$ and $R=K_1$ then

$$f(G_{2k-1}, \lambda)=(-1)^k L_k(\lambda^2)$$

and

$$f(G_{2k}, \lambda)=(-1)^k \lambda L_k^1(\lambda^2)$$

for $k=1, 2, \ldots$, where $L_k(\lambda)$ and $L_k^s(\lambda)$ denote the Leguerre and the generalized Laguerre polynomial, respectively ([6] pp. 45 and 52).

The above two statements have an important consequence.

Propositon 5. The Hermite and the Laguerre polynomials are both $\Phi$-graphic and $\alpha$-graphic.

Another interesting special situation arizes when $d_1=d_2=\cdots=d_m=d$. Then the relation (8) yields

$$f(G_m)=A\,((\lambda+\sqrt{\lambda^2-4d})/2)^m+B\,((\lambda-\sqrt{\lambda^2-4d})/2)^m,$$

which after the substitution $\lambda=2\sqrt{d}\cos t$ becomes

$$f(G_m)=d^{m/2}\,[(A+B)\cos mt+i\,(A-B)\sin mt]$$

and by taking into account the initial conditions (9) one arrives to the following result.

Proposition 6. Id $d=(d, d, \ldots, d)$ and $\lambda=2\sqrt{d}\cos t$, then

$$f(G_m)=d^{m/2}\left[\Phi(R)\frac{\sin(m+1)t}{\sin t}-\sqrt{d}\,\Phi(R-v_0)\frac{\sin mt}{\sin t}\right].$$

For $d=1$ the above formula reduces to a previously obtained result (corollary 3.1 in [2]).

Proposition 7. If $d=(d, d, \ldots, d)$, $R=K_1$ and $\lambda=2\sqrt{d}$co $t$, then

$$f(G_m)=d^{(m+1)/2}\frac{\sin(m+2)t}{\sin t}.$$

Therefore, the zeros of the cheracter:st'c (and match'ng) polynomial in this latter case are the numbers $2\sqrt{d}\cos\frac{j\pi}{m+2}$, one times for $k=1, 2, \ldots, m+1$ and $2\sqrt{d}\cos\frac{k\pi}{j+2}$, $(d-1)\,d^{m-j-1}$ times for $k=1, 2, \ldots, j+1$ and $j=0, 1, \ldots, m-1$.

If $d=1$, Proposition 7 reduces to the well known characteristic polynomial and spectrum of the path $P_{m+1}$ w th $m+1$ vertices.

Conclud ng the pre ent cons derations, we would like to state our main result, wh ch, however, is just a reformulation of Proposition 2.

Theorem. *Let G be an arbitrary graph and* $v_0$ *its arbitrary vertex. Let* $d_k$ $(k=1, 2, \ldots)$ *be arbitrary positive integers. Then the polynomials* $f_k = f_k(\lambda)$ *defined recursively as*

$$f_k = \lambda f_{k-1} - d_k f_{k-2}; \quad f_0 = \Phi(G); \quad f_{-1} = \Phi(G - v_0)$$

*are* $\Phi$*-graphic. The polynomials* $g_k = g_k(\lambda)$ *defined recursively as*

$$g_k = \lambda g_{k-1} - d_k g_{k-2}; \quad g_0 = \alpha(G); \quad g_{-1} = \alpha(G - v_0)$$

*are* $\alpha$*-graphic.*

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Faculty of Science
Universitty of Kragujevac
34000 Kragujevac
P. O. Box 60