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ON THE BROWN — McCOY RADICAL OF GROUP RINGS

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Abstract

We show if R is a simple ring with identity and G a finitely generated torsion free Abelain group, then the group ring RG is Brown-McCoy semisimple. We also prove that RG is Brown--McCoy semisimple if R is Brown-McCoy semisimple and where G is a finitely generated torsion free Abelian group.

Terminologies undefined here will have the same meaning as in [1]. If R is a ring with identity, we denote the Brown-McCoy radical of R by $B(R)$.

Proposition 1. *If R is a simple ring with identity and $G=\langle x\rangle$, the infinite cyclic group generated by x, then RG is a principal ideal ring.*

Proof. Let A be any ideal in RG. Since $R[x]$ is a subring of RG, $A\cap R[x]$ is an ideal in $R[x]$. We can pick a nonzero element $a(x)\in A\cap R[x]$ with minimal degree. Since the leading coefficients of all the elements of $A\cap R[x]$ with minimal degree, (say n), together with (0), forms an ideal in R, and R is a simple ring with identity, we can without loss of generality, assume that $a(x)$ is monic. Firstly, we show that $A\cap R[x]=\langle a(x)\rangle$. To this purpose we prove that $a(x)R[x]=R[x]a(x)=\langle a(x)\rangle$. Let $r\in R$, then $(a(x)r-ra(x))\in A\cap R[x]$ and degree $(a(x)r-ra(x))<n$. Hence $a(x)=ra(x)$ for every $r\in R$ and consequently $a(x)R[x]=R[x]a(x)$. From this and the definition of $\langle a(x)\rangle$ it follows that $\langle a(x)\rangle=a(x)R[x]=R[x]a(x)$. Let $f(x)$ be an arbitrary element of $A\cap R[x]$ of degree k with leading coefficient β. If $n=k$, then $[a(x)\beta-f(x)]\in A\cap R[x]$ and degree $(a(x)\beta-f(x))<n$. Consequently $f(x)= -a(x)\beta\in a(x)R[x]=\langle a(x)\rangle$. Suppose that every element of $A\cap R[x]$ of degree $k, n\leq k\leq m$, is an element $\langle a(x)\rangle$, then if $f(x)$ is of degree $m+1$ we have $g(x)=f(x)-a(x)\beta x^{m+1-n}\in A\cap R[x]$ with degree $g(x)\leq m$. From our assumption there exists $h(x)\in R[x]$ such that $g(x)=a(x)h(x)$ and consequently $f(x)=a(x)p(x)$, where $p(x)=h(x)\beta x^{m+1-n}\in R[x]$. Hence $A\cap R[x]\subseteq\langle a(x)\rangle$. However, since $A\cap R[x]$ is an ideal in $R[x]$ we have $\langle a(x)\rangle\subseteq A\cap R[x]$. Consequently $A\cap R[x]=\langle a(x)\rangle$. We claim that $A=a(x)RG=RGa(x)$ Clearly $a(x)RG\subseteq A$. Next, let $y\in A$, $y\neq 0$. We can write $y=x^j f(x)$ for some integer j and $f(x)\in R[x]$. Then $yx^{-j}=f(x)\in A\cap R[x]$. Hence $f(x)\in\langle a(x)\rangle$ and we can write $f(x)=a(x)k(x)$ where $k(x)\in R[x]$. Hence $y=f(x)x^j= =a(x)k(x)x^j\in a(x)RG$. Therefore, $a(x)RG=RGa(x)=A$. Thus we have proved that A is a principal ideal in RG, generated by $a(x)$. $\square$

Lemma 2. *Let R be a simple ring with identity and G an infinite cyclic group. Then $B(RG)=(0)$.*

Proof. Let $G=\langle x\rangle$ be the infinite cyclic group generated by x. Suppose now I is the Brown-McCoy radical of RG. From Proposition 1 there exists a monic polynomial $a(x)$ of degree n, say, in $I\cap R[x]$ such that $I=a(x)RG=RGa(x)$. Then $I=\langle a(x)\rangle$ and $a(x)$ is $G-$ regular in RG, that is

$$a(x)\in G(a(x))=\{a(x)y-y+\Sigma(g_i a(x) h_i-g_i h_i)\}$$

where the summation is over a finite range and y, g_i, $h_i\in RG$. Since $a(x)RG=$ $=RGa(x)$ we have $G(a(x))=F(a(x))=\{a(x)y-y)\}$. Hence there is $s\in RG$, $s\neq 0$, such that $a(x)s-a(x)-s=0$. By comparing degrees we see that either degree $s=0$ or degree $a(x)=0$. If degree $s=0$ but degree $a(x)\neq 0$ then, for the coefficient of x^n in $a(x)s$ and $a(x)$ to cancel, we must have $s=1$. This is impossible for it will imply $1=0$. Similarly we can prove that neither degree $s=0$ and degree $a(x)=0$ nor degree $s\neq 0$ and degree $a(x)=0$. Hence $s=0$ and hence $a(x)=0$. Consequently $I=(0)$, i.e. $B(RG)=(0)$. □

Lemma 3. *Let R be a ring with identity and G an infinite cyclic group. If $B(R)=(0)$ then $B(RG)=(0)$.*

Proof. Since $B(R)=(0)$, it follows from [1], Theorem 7.26 that $\bigcap\limits_{i\in U} M_i=(0)$ where $\{M_i : i\in U\}$ is the family of all the modular maximal ideals in R. Hence for each $i\in U$, R/M_i is a simple ring with identity. Now $RG/M_i(G)\cong(R/M_i)G$ and from Lemma 2 $B(R/M_i(G)=(0)$ for each i. Put $R/M_i=\bar{R}_i$. From [1] Theorem 7.27 it now follows that for each $i\in U$, $\bar{R}_i G$ is a subdirect sum of simple rings with unity. Furthermore, $\bigcap\limits_{i\in U}(M_i G)=(\bigcap\limits_{i\in U} M_i)G=(0)$ and consequently it follows from [1], Theorem 3.9 that RG is isomorphic to a subdirect sum of the rings $\bar{R}_i G$. Hence RG is isomorphic to a subdirect sum of simple rings with unity and consequently $B(RG)=(0)$. □

Theorem 4. *If R is a simple ring with identity and G finitely generated torsion free Abelian group, then $B(RG)=(0)$.*

Proof. Indeed, since G is a finitely generated torsion free Abelian, then $G\cong C_1\times C_2\times\cdots\times C_n$, where C_i is infinite cyclic. But then $RG\cong(RC_1)(C_2\times\times\cdots\times C_n)$, thus we may apply Lemmas 2 and 3 and induction to complete the proof. □

Theorem 5. *Let R be a ring with identity and G a finitely generated torsion free Abelian group. If $B(R)=(0)$ then $B(RG)=(0)$.*

Proof. Put $G=C_1\times C_2\times\cdots\times C_n$, C_i infinite cyclic. Then the result follows by Lemma 3 and induction. □

Corollary 6. *Let R be any ring with identity and G a finitely generated torsion free Abelian group. Then $B(RG)\subseteq B(R)G$.*

Proof. Consider the isomorphism $[R/B(R)]G \cong RG/B(R)G$. Since for any ring, $B(R)$ is the smallest ideal K of R such that $B(R/K)=(0)$, it follows from Theorem 5. that $B(RG) \subseteq B(R)G$. □

REFERENCES

[1] N. H. McCoy, *The Theory of Rings*, MacMillan, New York, 1965.

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