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ON RELATIONS BETWEEN LOCAL AND GLOBAL MONOTONY OF MAPPINGS OF ORDERED SETS

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0. Let $(S, \leqslant)$ and $(T, \leqslant_1)$ be odered sets (partially or totally) and let f be a mapping of S into T. As usually, we say that the mapping f is *increasing* on S if for all $x, y \in S$

$$x < y \Rightarrow f(x) \leqslant_1 f(y),$$

and that it is *strictly increasing* on S if for $x, y \in S$

$$x < y \Rightarrow f(x) <_1 f(y),$$

and we accept also the corresponding definitions of *decreasing* and *strictly decreasing mapping*. The increasing and decreasing mappings are called *monotonic* and strictly increasing or decreasing *strictly monotonic mappings* (*functions*). The importance of monotonic functions in certain domains is well-known; for example, in the cases of real functions and real sequences. However, in last decades the more general types of such functions have been studied very much; in particular, after the well-known fundamental result of *A*. Tarski [1] concerning fixed points of increasing mappings. Numerous papers are dedicated to various generalizations of this result and to other applications of increasing and decreasing mappings of ordered sets.

The subject of this paper is the question under which conditions the "local monotony", or the "local strict monotony", i. e. the (strict) monotony on at least one interval containing x, for any $x \in S$, or some analoguous property, implies the "global (strict) monotony", i. e. the (strict) monotony on the whole set S. In some simple and familiar cases, as in the cases of real sequence and real function defined on an interval, this implication is intuitively evident and can easily be rigorously proved; but it is impossible to extend this intuition to some more complicate cases, as it is not difficult to show by examples. Our results contained in Theorems 1 – 3 and in supplementary statements give different conditions under which a property apparently weaker implies the monotony, resp. the strict monotony, on the whole domain of the consiredred function, and establishes that any of these conditions cannot be omitted or weakened, in a definite sense, and moreover (Theorems 1 and 3) that a characteristic condition is also necessary for the validity of the implication in question, all other conditions being unchanged. We add to these general results a characterization of monotony and of strict monotony of a real function defined on an interval (without any other suppositions) in terms of upper and lower, left and right derivative (Theorem 4), which we deduce from Theorem 1.

We point out that every of Theorems $1-3$ and $1'$ is really *twofold*: namely, it contains two statements condensed in one statement; the first of them consists of the text and the symbols out of parentheses and in the second one the text and the symbols in parentheses everywhere replace the corresponding elements out of them. We note also that, for $x, y \in S$ and $x \prec y$, the set $[x, y[$ is defined by

$$[x, y[\stackrel{\text{def}}{=} \{t : t \in S \wedge x \preccurlyeq t \prec y\},$$

and similarly the sets $]x, y]$, $]x, y[$ and $[x, y]$ are defined.

1. The following results, as formulated, refer to (strictly) increasing mappings. From every of them one can obviously, by suitable changes, obtain the corresponding statement concerning (strictly) decreasing function, and also the statement in which the increasing function is retained, but otherwise the left and right sides change their roles.

We start from a characterization of conditional completness of ordered sets, in some sense ccmparable to the characterization of complete lattice given by Anne Devis in [2] (as a supplementary result to the cited paper of Tarski).

Theorem 1. *Let us suppose that*

(1) $(S, \preccurlyeq)$ *is a chain* (*totally ordered set*).

Then the condition

(2) $\left\{ \begin{array}{l} (S, \preccurlyeq) \textit{ is conditionally complete (that is, every nonempty} \\ \textit{subset of } S \textit{ with upper bound has its supremum)} \end{array} \right.$

is necessary and sufficient for the validity, whenever

(3) $(T, \preccurlyeq_1)$ *is an ordered set*

and $f : S \to T$, *of the implication*:

(4) $$(\forall x \in S) \left\{ \begin{array}{l} (\forall y \in S)\,(y \prec x \;\Rightarrow\; (\exists t \in [y, x[\,) f(t) \preccurlyeq_1 f(x)) \wedge \\ (x \textit{ is not } \max S \;\Rightarrow\; (\exists y \in S)\,(x \prec y \wedge (\forall t \in]x, y]) f(x) \preccurlyeq_1 f(t))) \end{array} \right.$$
$$\Rightarrow \qquad\qquad (\prec_1)$$

(5) f *increases* (*strictly increases*) *on* S.

Proof. 1° *Sufficiency*. Under the suppositions (1), (2) and (3), let (4) be satisfied. Suppose that (5) is not true, i. e. that there exist $x \in S$ and $y \in S$ such that

$$x \prec y \wedge \neg f(x) \preccurlyeq_1 f(y)$$
$$(\prec_1)$$

($\neg$ denotting the negation). Then obviously x is not the maximum of S. This and (4) imply that the set

(6) $$P_x \stackrel{\text{def}}{=} \{t : t \in S \wedge x \prec t \wedge (\forall u \in]x, t]) f(x) \preccurlyeq_1 f(u)\}$$
$$(\prec_1)$$

is nonempty. By (1), y is an upper bound of P_x and so, by (2), there exists

$$z = \sup P_x. \tag{7}$$

On account of (6), $x \prec z$. We have further

$$(\forall\, t \in]x, z[)\, f(x) \underset{(\prec_1)}{\preccurlyeq_1} f(t). \tag{8}$$

Indeed, if $t \in]x, z[$, then, by (7), there exists $u \in]t, z] \cap P_x$, which implies, accordingly to (6), $f(x) \underset{(\prec_1)}{\preccurlyeq_1} f(t)$. If $]x, z[= \varnothing$, then we conclude, using (6), (7) and the fact $P_x \neq \varnothing$, that

$$z \in P_x. \tag{9}$$

If $]x, z[\neq \varnothing$, let $u \in]x, z[$. Then $[u, z[\subset]x, z[$ and, by (4), there exists $t \in [u, z[$ such that $f(t) \preccurlyeq_1 f(z)$. So

$$(\exists\, t \in]x, z[)\, f(t) \preccurlyeq_1 f(z). \tag{10}$$

From (8) and (10)

$$f(x) \underset{(\prec_1)}{\preccurlyeq_1} f(z)$$

follows, which togheter with (8) implies (9) again.

The proved relation (9) and the fact that y is an upper bound of P_x which does not belong to P_x imply $z \prec y$, and so z is not $\max S$. Hence, by (4), there is a $u \in S$ such that $z \prec u \wedge (\forall\, v \in]z, u])\, f(z) \underset{(\prec_1)}{\preccurlyeq_1} f(v)$. Then, accordingly to (9), $u \in P_x$, and $z \prec u$, in contradiction with (7). The proof of sufficiency is over.

2° *Necessity.* Suppose that the chain $(S, \preccurlyeq)$ is not conditionally complete. Then there exists a nonempty part U of S which is bounded from above and has not its supremum. Let us denote by B the set of all upper bounds of U and put $A = S \setminus B$. The sets A and B are nonempty, A is without maximum and B without minimum and we also have

$$(\forall\, x \in A)\, (\forall\, y \in B) \quad x \prec y.$$

Let us put $T = S$ and

$$(\forall\, x, y \in T) \begin{cases} x \preccurlyeq_1 y \overset{\text{def}}{\Leftrightarrow} x \preccurlyeq y, \text{ if } x, y \in A \text{ or } x, y \in B, \\ x \prec_1 y, \text{ if } x \in B \text{ and } y \in A. \end{cases}$$

It is easy to see that this defines the ordering relation $\preccurlyeq_1$ in T.

Then the mapping $f: S \to T$ defined by $f(x) = x\ (x \in S)$ satisfies the condition (4), with symbols $\prec$ and $\prec_1$ (and even a stronger condition), and does not insrease on S. This completes the proof of necessity.

It is clear that Theorem 1 contains the following statement:

Theorem 1′. *Under conditions* (1), (2), (3) *and* (4), *for every* $f: S \to T$, (5) *holds.*

Any of conditions (1), (2) and (4) cannot be omitted or replaced by a definite weaker condition, even if some other conditions are simultaneously, in a definite manner, strenghtened. More precisely, we have the following supplementary.

P r o p o s i t i o n 1. *In Theorem* $1'$ *one cannot:*

1° *omit condition* (1) (*i. e. replace it by the supposition that* $(S, \preccurlyeq)$ *is only an ordered set*), *even if* (3) *is simultaneously replaced by the condition*

$$(11) \qquad (T, \preccurlyeq_1) \text{ is a chain}$$

and, eventually, the condition (4) *by the following*

$$(12) \quad (\forall x \in S)(\exists y \in S)(\exists z \in S)(y \prec x \prec z \wedge f \text{ strictly increases on } [y, z]);$$

2° *omit condition* (2), *even if at the same time one requires the total ordering of* $(T, \preccurlyeq_1)$ *and one replaces* (4) *by* (12);

3° *replace* (4) *by the condition in which the conjunction contained in* (4) *is reduced to the first or to the second of its parts, even if the retained part is simultaneously replaced by a stronger condition, namely the first one by*

$$(\exists y \in S)(y < x \wedge f \text{ stictly increases on } [y, x]),$$

and the second one by

$$(\exists y \in S)(x \prec y \wedge f \text{ strictly increases on } [x, y]),$$

adding simultaneously the condition (11);

4° *replace* (4) *by*

$$(\forall x, y, z \in S)((y \prec x \Rightarrow (\exists t \in [y, x[)\, f(t) \prec_1 f(x)) \wedge (x \prec z \Rightarrow$$
$$\Rightarrow (\exists u \in]x, z])\, f(x) \prec_1 f(u))).$$

P r o o f. 1° Let the set S be formed of the followed two chains:

$$0 \prec \ldots \prec -3 \prec -2 \prec 1 \prec 3 \prec \ldots \text{ and } \ldots \prec -4 \prec -2 \prec 0 \prec 2 \prec 4 \prec \ldots,$$

each element different from zero of the first chain being non comparable to any element $\succ 0$ of the second chain. Let further $(T, \preccurlyeq_1) = (Z, \leqslant)$, where Z denotes the set of all integers; $f(k) = k \; (k \in S = Z)$. In this case the conditions of Theorem $1'$ with all changes mentionned in 1° (and also with the condition (4) unchanged) hold, but (5) does not hold, because $0 \prec -1$, $f(0) = 0 > -1 = f(-1)$.

2° The assertion is proved by the following example:

$$S = T = R \setminus \{0\}, \; \preccurlyeq \, = \, \preccurlyeq_1 \, = \, \leqslant, \quad f(x) = -\frac{1}{x} \; (x \in S)$$

(R denotes the set of all real numbers).

3° α) *Reduction to the first part of the conjunction, this part being strenghtened.* Example:

$$(S, \preccurlyeq) = (T, \preccurlyeq_1) = (R, \leqslant), \quad f(x) = \begin{cases} x, & x \leqslant 0 \\ -\dfrac{1}{x}, & x > 0. \end{cases}$$

β) *Reduction to the second part of the conjunction, this conjunction being strengthened.* Example:

$$(S, \preccurlyeq) = (T, \preccurlyeq_1) = (R, \leqslant), \quad f(x) = \begin{cases} -\dfrac{1}{x}, & x<0 \\ x, & x \geqslant 0. \end{cases}$$

4° Example: $(S, \preccurlyeq) = (T, \preccurlyeq_1) = (R, \leqslant)$,

$$f(x) = \begin{cases} 0, \text{ if } x \text{ is irrational or } x=0, \\ q, \text{ if } x = \dfrac{p}{q},\ p \in Z \setminus \{0\},\ q \in N, \text{ the fraction } |p|/q \text{ being irreducible, and } q \text{ is even}, \\ -q, \text{ if } \ldots\ldots\ldots\ldots, \text{ and } q \text{ is odd.} \end{cases}$$

(We remark that this function f has the properties $D^- f(x) = D^+ f(x) = +\infty (x \in R)$ and $D_- f(x) = D_+ f(x) = -\infty\ (x \in R)$.) –

However, there are some other and to Theorem 1 incomparable statements which also give sufficient conditions for the monotony, resp. the strict monotony, of the mapping $f: S \to T$. First, we formulate the folowing result, *in which the total ordering of $(S, \preccurlyeq)$ is not supposed.*

Theorem 2. *Let the following conditions be satisfied:*

(13) *every nonempty subset of S bounded from below has a minimal element and*

(14) $$(\forall x \in S)(\forall y \in S)\,(y \prec x \Rightarrow (\exists t \in [y, x[\,) f(t) \underset{(\prec_1)}{\preccurlyeq_1} f(x)).$$

Then the mapping f is increasing (stictly increasing) on S.

Proof. Let the conditions (13) and (14) be satisfied and suppose that f is not increasing (strictly increasing) on S. Then there exist $x \in S$ and $y \in S$ such that

(15) $$x \prec y \wedge \neg f(x) \underset{(\prec_1)}{\preccurlyeq_1} f(y).$$

For such a fixed x, let us denote by Q_x the set of all y satisfying (15). The set Q_x is bounded from below and consequently has a minimal element z. We have $x \prec z$; by (14), there exists $t \in [x, z[$ such that

(16) $$f(t) \underset{(\prec_1)}{\preccurlyeq_1} f(z).$$

We also have

(17) $$f(x) \preccurlyeq_1 f(t).$$

Indeed, (17) holds if $x = t$; otherwise, $x \prec t \prec z$ and hence $f(x) \underset{(\prec_1)}{\preccurlyeq_1} f(t)$, since z is a minimal element of Q_x. By (16) and (17), $f(x) \underset{(\prec_1)}{\preccurlyeq_1} f(z)$, what contradicts the supposition that z is a minimal element of Q_x. This contradiction proves the theorem. –

One can add the following supplementary

Proposition 2. *In Theorem* 2:

1° *condition* (13) *cannot be omitted, even if one supposes simultaneously the total ordering of* $(S, \leqslant)$ *and* $(T, \leqslant_1)$, *and one replaces* (14) *by the condition* (12);

2° *condition* (13) *cannot be replaced by the condition* (2), *even if one supposes simultaneously the total ordering of* $(S, \leqslant)$ *and* $(T, \leqslant_1)$ *and one replaces* (14) *by*

$$(18)\qquad (\forall x \in S)\ (x \textit{ is not } \min S \Rightarrow (\exists y < x)\ (\forall t \in [y, x[\ )\ f(t) \underset{(<_1)}{\leqslant_1} f(x)).$$

This statement follows immediately from Proposition 1.

Using the previous theorem and adding a supplementary consideration, one easily proves the

Theorem 3. *The condition*

(13') *every nonempty subset of S bouned from below has a min mum*

is necessary and sufficient for the validity, whenever $(S, \leqslant)$ *is a chain, of the implication*

$$(18) \Rightarrow (5).$$

Proof. The sufficiency follows immediately from Theorem 2, since (13') implies (13) and in the case when $(S, \leqslant)$ is a chain (18) implies (14).

In order to prove the necessity, let us suppose that $(S, \leqslant)$ is a chain and that the set $C \subset S$ is bounded from below and has not its minimum. Let us denote the set of all lower bounds of C by A and let $B = S \setminus A$. If $T = S$, and $\leqslant_1$ is defined as in the second part of the proof of Theorem 1 and $f(x) = x\ (x \in S)$, then (18) holds and (5) does not. –

A remark concerning Theorem 1 and Theorem 1'. It is easy to see that these theorems remain true if one displaces in (4) the symbol $(<_1)$ on the corresponding place under the first row.

2. On the basis of Theorem 1, we shall prove the following statement characterizing, in the general case, the strict increase of a real function defined on an interval.

Theorem 4. *Let* $f: I \to R$, *where R denotes the set of all real numbers and* $I (\subset R)$ *a nonempty interval.*

1° *For the increase of f on I it is necessary that*

$$(19')\qquad \begin{cases} D_- f(x) \geqslant 0 \wedge D_+ f(x) \geqslant 0 \textit{ on } I, \\ \textit{excluding the requirement that } D_- f(a) \geqslant 0 \textit{ if } a = \inf I \in I, \textit{ and that} \\ D_+ f(b) \geqslant 0 \textit{ if } b = \sup I \in I, \end{cases}$$

and is sufficient that

$$(19) \qquad \begin{cases} D^- f(x) \geqslant 0 \wedge D_+ f(x) \geqslant 0 \ \textit{on } I, \\ \textit{with a restriction similar to that included in } (19'). \end{cases}$$

Therefore any of conditions (19′) *and* (19) *is necessary and sufficient for the increase of f on I (and so* (19′) $\Leftrightarrow$ (19)).

2° *A necessary and sufficient condition for the strict increase of f on I is*

$$(19) \wedge (20) \ (\textit{or } (19') \wedge (20)),$$

where

(20) $(\forall x \in S)\ (D^- f(x) > 0 \vee D^+ f(x) > 0)$, *the set $S \subseteq I$ being dense on I.*

Remark 1. One can formulate the corresponding similar theorem concerning strict decrease. Also, in the condition (19) of Theorem 4 one can replace $D^- f(x) \geqslant 0 \wedge D_+ f(x) \geqslant 0$ by $D_- f(x) \geqslant 0 \wedge D^+ f(x) \geqslant 0$. Both facts are clear with respect to the following proof and to the remarks at the beginning of **1**.

Remark 2. Another statement containing sufficient condition, expressed partially by unilateral upper derivative, for the increase of a real function on an interval is well-know (see, for instance, [3], pp. 354—355, Example IV). This condition is the conjunction of the nonnegativity of $D^+ f$, or of $D^- f$, on the interval and the continuity of f on the same interval. It is evident that this condition is essentially stronger than the corresponding condition (19) in Theorem 4.

Proof of Theorem 4. 1° It is evident that the condition (19′) is necessary for the increase of f on I. Let us suppose the condition (19) be satisfied. Then, for each $\varepsilon > 0$, the fonction $g_\varepsilon : I \to R$ defined by

$$g_\varepsilon(x) = f(x) + \varepsilon x \qquad (x \in I)$$

has the property

$$(\forall x \in I)\ (D^- g_\varepsilon(x) = D^- f(x) + \varepsilon > 0 \wedge D_+ g_\varepsilon(x) = D_+ f(x) + \varepsilon > 0).$$

This implies that the function g_ε satisfies the conditions of Theorem 1′, with symbols in parentheses. Hence, for each $\varepsilon > 0$, g_ε increases strictly on I, and consequently we have, for fixed $x, y \in I$ such that $x < y$,

$$f(x) + \varepsilon x < f(y) + \varepsilon y \qquad (\varepsilon > 0).$$

Making $\varepsilon \to +0$, one gets $f(x) \leqslant f(y)$ $(x, y \in I,\ x < y)$. Therefore, the condition (19) is sufficient.

2° If the condition $(19) \wedge (20)$ is satified, then in the first place f increses on I. Further, for $x, y \in I$ and $x < y$, there exists $z \in]x, y[$ such that $D^- f(z) > 0$ or $D^+ f(z) > 0$. Let, for instance, the first inequality hold. Then we have, for a $t \in]x, z[$, $(f(t) - f(z))/(t - z) > 0$, i.e. $f(t) < f(z)$; so we get $f(x) \leqslant f(t) < f(z) \leqslant f(y)$ and consequently $f(x) < f(y)$. If $D^+ f(z) > 0$,we similarly obtain the same conclusion. So the conjunction $(19) \wedge (20)$ iz sufficient for the strict increase of f on I. It is also necessary, since (19) is already necessary for the increase of f on I, and if (19) is satisfied and (20) is not, then f increases on I and there exists an interval $(\alpha, \beta) \subset I$ such that

$$(\forall x \in (\alpha, \beta))\ D^- f(x) = D_- f(x) = D^+ f(x) = D_+ f(x) = 0,$$

which implies (for instance, by 1° and by the corresponding statement concerning decrease of f on I; see Remark 1) $f(x)=\text{const}$ on (α, β).

An immediate corollary (special case) of the preceding statement is

Theorem 4′. *Let the real function f have its derivative on the interval I, understanding by the existence of the derivative at $a=\min I$ the existence of $f_+'(a)$ and similarly for $b=\max I$. (The derivative can be finite or infinite and the continuity of f on I is not supposed.) Then:*

1° *the condition*

$$f'(x)\geqslant 0 \text{ on } I \tag{21}$$

is necessary and sufficient for the increase of f on I;

2° *f increases strictly on I if and only if* (21)$\wedge$(22) *holds, where*

$$(\forall x\in S) f'(x)>0, \; S\subset I \text{ being dense on } I. - \tag{22}$$

At the end, let us remark that in textbooks exposing foundations of mathematical analysis the connexion between monotony and sign of derivative is usually established in the form of a statement which supposes the continuinuity of the function and the existence of its derivative (finite or infinite) on the considered interval. Its proof is usually based on Lagrange's or Rolle's mean value theorem, whose proof uses Weierstrass' theorem on extrema of a continuous function on a segment. Theorem 4′ gives an alternative possibility in this way. Namely: Theorem 4′ is more general than the usual theorem, and its direct proof (that is the appropiate form of the combination of the proofs of Theorems 1′ and Theorem 4) uses neither properties of continuous functions nor mean value theorems, and even not the notion of continuous function (it really uses only the conditional completness of $(R, \leqslant)$), and finally it is not longer than the proof of the usual theorem, taking into consideration all auxiliary results preceding this second proof.

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