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A NOTE ON A GENERALIZATION OF SOME UNDECIDABILITY RESULTS IN GROUP THEORY

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In M. Rabin's fundamental paper [1] it was proved that there is no algorythm to decide for every given finite presentation Π whether the group G_{Π} determined by that presentation is decomposable into

1° a nontrivial direct product, or into

2° a nontrivial free product, or else into

3° a free product of finite groups.

The result 3° (Theorem 2.11 in [1]) is the solution of the problem posed originally by S. Kochen. One is of course immediately led to pose a number of analogous problems concerning free products of Abelian, cyclic, free, torsion-free, solvable, simple etc. groups.

In [2] an efficient test was derived to characterize Markov properties of finitely presentable (f.p.) groups (a property P is a Markov property if it is algebraic and nontrivial, and if there exists a group K which cannot be embedded into a group enjoying P). Making use of this test we are now in position to resolve all of the above problems, augmenting thus the claim of its applicability. Furthermore, some of the similar problems concerning decidability of decomposability into a diιect product can also be settled in this way.

In the sequel, let $\mathcal{K}$ denote the class of all f.p. groups, and $\mathcal{U}$ the class of all universal f.p. groups (i.e those f.p. groups that contain isomorphic copies of all the f.p. groups).

Lemma: *There is no universal f.p. group U decomposable into a free product of non-universal f.p. groups, i.e.*

$$\neg(\exists U\in\mathcal{U})(\exists n\in N)(\exists G_1, \ldots, G_n\in\mathcal{K}\setminus\mathcal{U})\, U\cong G_1*\cdots*G_n$$

P r o o f: Let G_1 and G_2 be non-universal groups, and suppose groups H_1 and H_2 fulfill

$$\neg(\exists H_1')(H_1' < G_1 \wedge H_1' \cong H_1)$$

$$\neg(\exists H_2')(H_2' < G_2 \wedge H_2' \cong H_2)$$

(i.e. H_1, H_2 are not embedable into G_1, G_2 respectively).

Let further

$$G = G_1 * G_2.$$

According to Kurosh's theorem on the subgroups of a free product [3], each subgroup H of G is of the form

$$H \cong F * \prod_i{}^* G_i'$$

where F is a free group and G_i' are groups conjugated in G to subgroups of G_1 and G_2. Thus, H is either a trivial (i.e. of order one) group or a free group or else a free product of some subgroups of G_1, G_2 and perhaps of F. Hence, if $H_1 \times H_2$ is (isomorphic to) a subgroup of G, then $H_1 \times H_2$ is a free group or a free product of some groups. But these both are impossible. Indeed, a free group is not decomposable into a nontrivial direct product; also, none of the groups are simultaneously decomposable into a nontrivial direct and free product (in [3], Theorem of Baer R. and Levi F.). So, group $H_1 \times H_2$ is not embeddable into G, and G is not a universal group. ▌

T h e o r e m: *Let P be an arbitrary Markov property of f.p. groups. Then there is no effective algorythm to decide for every finite presentation Π whether the group G_Π can be decomposed into a free product of groups enjoying the property P.*

P r o o f: Let P be a Markov property as required and let R_P denote the property of "being decomposable into a free product of groups enjoying P", i.e.

$$(*) \qquad R_P(G) \Leftrightarrow (\exists n \in N)(\exists H_1, \ldots, H_n)(P(H_1) \wedge \cdots \wedge \wedge P(H_n) \wedge G \cong H_1 * \cdots * H_n).$$

According to the mentioned test (Theorem 1 in [2]), for a nontrivial algebraic property P one has

$$\mathcal{M}(P) \Leftrightarrow (\forall U \in \mathcal{U}) \neg P(U)$$

i.e. P is a Markov property iff none of the universal groups enjoys P.

Hence, groups $H_1, \ldots, H_n$ in $(*)$ are not universal. Using the preceeding Lemma, each group G enjoying R_P is also a non-universal one. Making once more use of [2], one concludes that R_P is a Markov property, and hence (using [1]) algorythmicaly unrecognizable. ▌

Now, a next natural step is to extend the investigation to other group construction, first of all being the direct product. By analogous arguments one can prove the following:

One cannot algorythmicaly recognize the properties of being decomposable into a (finite) d i r e c t product of finite, Abelian, torsion-free, solvable, etc. groups.

Namely, the product groups obtained by finite direct products are also finite, Abelian, ..., etc. respectively, and hence non-universal themselves.

But the reader should note that it remains still open whether one can extend this to every Markov property P, i.e. whether the analogue of the Theorem given above holds for the direct product also.

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[3] A. G. Kurosh: *Theory of Groups*, Chelsea, New York, 1955.