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AN ESTIMATE OF THE RATE OF CONVERGENCE FOR FOURIER SERIES OF FUNCTIONS OF BOUNDED VARIATION

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Let f be a 2π-periodic function of bounded variation on $[-\pi, \pi]$ and let $S_n(f, x)$ be the n-th partial sum of the Fourier series of f. If

$$g_x(t)=f(x+t)+f(x-t)-f(x+)-f(x-)$$

then

$$S_n(f, x)-\frac{1}{2}(f(x+)+f(x-))=\frac{1}{\pi}\int_0^{\pi} g_x(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\,dt \tag{1.1}$$

and the well-known theorem of Dirichlet-Jordan (see [1], vol. I, p. 57) states that

$$\lim_{n\to\infty}\left(S_n(f, x)-\frac{1}{2}(f(x+)+f(x-))\right)=0. \tag{1.2}$$

The aim of this note is to give a quantitative version of Dirichlet-Jordan's theorem by showing that, for $n\geqslant 1$,

$$\left|S_n(f, x)-\frac{1}{2}(f(x+)+f(x-)\right|\leqslant\frac{3}{n}\sum_{k=1}^{n}V_0^{\pi/k}(g_x) \tag{1.3}$$

where $V_0^y(g_x)$ is the total variation of g_x on $[0, y]$, $y\in[0, \pi]$.

Since $g_x(t)$ is a function of bounded variation on $[0, \pi]$, continuous at the point $t=0$, the total variation $V_0^y(g_x)$, $y\in[0, \pi]$, of g_x, is a continuous function at $y=0$ and consequently

$$V_0^{\pi/k}(g_x)\to 0 \qquad (k\to\infty).$$

This implies that the right-hand side of inequality (1.3) also converges to 0 and (1.2) follows.

A result related to (1.3) was obtained by G. I. Natanson [2] who proved that for 2π-periodic continuous functions of bounded variation on $[-\pi, \pi]$ we have

$$(1.4)\qquad |S_n(f, x)-f(x)|\leqslant \frac{2}{\pi n}\int_{4/\pi n}^{\pi} w_{V(f)}(2t)\, t^{-2}\, dt+\frac{1}{\pi n} V_{-\pi}^{\pi}(f).$$

Here, $w_{V(f)}(\delta)$ is the modulus of continuity of the total variation $V_{-\pi}^{t}(f)$, $t\in[-\pi, \pi]$, of f.

This result is a simple con.equence of (1.3) since for continuous functions of bounded variation we have

$$V_0^{\delta}(g_x)\leqslant V_{x-\delta}^{x+\delta}(f)\leqslant 2\, w_{V(f)}(\delta)$$

and (1.3) becomes

$$|S_n(f, x)-f(x)|\leqslant \frac{6}{n}\sum_{k=1}^{n} w_{V(f)}\left(\frac{\pi}{k}\right)$$

which is essentially equivalent to Natanson's result (1.4).

An estimate of a different nature for the rate of convergence of continuous functions of bounded variation was obtained by S. B. Stečkin [3] who proved that

$$|S_n(f, x)-f(x)|\leqslant M(f)\, w_f\left(\frac{1}{n}\right)\log\left(2+\frac{1}{w_f\left(\frac{1}{n}\right)}\right)$$

where $M(f)$ is a constant depending only on f and $w_f(\delta)$ is the modulus of continuity of f. A more precise version of Stečkin's result was obtained by V. G. Kominar [4].

For the proof of inequality (1.3) it is sufficient to show that the following result is true.

Theorem. *Let g be a 2π-periodic function of bounded variation on* $[0, \pi]$ *with $g(0)=0$. Then*

$$\left|\frac{1}{\pi}\int_0^{\pi} g(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\, dt\right|\leqslant \frac{3}{n}\sum_{k=1}^{n} V_0^{\pi/k}(g)$$

for $n=1, 2, \ldots$

Proof of the Theorem. We have

$$I_n(g)=\frac{1}{\pi}\int_0^{\pi} g(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\, dt =$$

$$=\frac{1}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right)g(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt=$$

$$=A_n(g)+B_n(g).$$

Since $\left|\dfrac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right|\leqslant n+\dfrac{1}{2}$ and $g(0)=0$ we have first

$$|A_n(g)|=\frac{1}{\pi}\left|\int_0^{\pi/n}(g(t)-g(0))\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt\right|\leqslant$$

$$\leqslant\frac{1}{\pi}V_0^{\pi/n}(g)\int_0^{\pi/n}\left(n+\frac{1}{2}\right)dt$$

or

$$|A_n(g)|\leqslant 2V_0^{\pi/n}(g). \tag{1.5}$$

To estimate $B_n(g)$ let

$$\wedge_n(x)=\int_x^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}dt.$$

We have then, by partial integration,

$$B_n(g)=\frac{1}{\pi}g\left(\frac{\pi}{n}\right)\wedge_n\left(\frac{\pi}{n}\right)+\frac{1}{\pi}\int_{\pi/n}^{\pi}\wedge_n(t)\,dg(t).$$

Since $g(0)=0$ and $|\wedge_n(x)|\leqslant\dfrac{\pi}{nx}$ for $0<x\leqslant\pi$, it follows that

$$|B_n(g)|\leqslant\frac{1}{\pi}V_0^{\pi/n}(g)+\frac{1}{n}\int_{\pi/n}^{\pi}\frac{1}{t}\,dV_0^t(g).$$

Using again integration by parts we find that

$$\int_{\pi/n}^{\pi}\frac{1}{t}\,dV_0^t(g)=\frac{1}{t}V_0^t(g)\Big|_{\pi/n}^{\pi}+\int_{\pi/n}^{\pi}V_0^t(g)\frac{dt}{t^2}.$$

Hence

$$|B_n(g)| \leqslant \frac{1}{n\pi} V_0^{\pi}(g) + \frac{1}{n} \int_{\pi/n}^{\pi} V_0^{t}(g) \frac{dt}{t^2}$$

or

$$|B_n(g)| \leqslant \frac{1}{n\pi} V_0^{\pi}(g) + \frac{1}{n\pi} \int_{1}^{n} V_0^{\pi/t}(g)\, dt.$$

Since $V_0^{\pi/t}(g)$ is a decreasing function, we have

$$\int_{1}^{n} V_0^{\pi/t}(g)\, dt \leqslant \sum_{k=1}^{n} V_0^{\pi/k}(g)$$

and so

$$\begin{aligned} |B_n(g)| &\leqslant \frac{1}{n\pi} V_0^{\pi}(g) + \frac{1}{n\pi} \sum_{k=1}^{n} V_0^{\pi/k}(g) \\ &\leqslant \frac{2}{n\pi} \sum_{k=1}^{n} V_0^{\pi/k}(g). \end{aligned} \tag{1.6}$$

Using (1.5) and (1.6) we find that

$$|I_n(g)| \leqslant 2 V_0^{\pi/n}(g) + \frac{2}{n\pi} \sum_{k=1}^{n} V_0^{\pi/k}(g) \leqslant \frac{3}{n} \sum_{k=1}^{n} V_0^{\pi/k}(g)$$

and the theorem is proved.

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