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INTEGRABILITY THEOREMS FOR JACOBI SERIES

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Our object here is to obtain results linking the integrability properties of a function with convergence properties of series involving its Fourier-Jacobi coefficients.

In § 1 we introduce the necessary notation and terminology. In § 2 we state our results (Theorems 1 and 2) and discuss their relationship with previous work. The proofs of Theorems 1,2 follow in §§ 3,4. We close in § 5 with some comments on related work.

§ 1. Preliminaries.

For $\alpha \geq \beta \geq -\frac{1}{2}$, we consider the probability measure $G^{(\alpha,\beta)}$ on $[-1, 1]$ defined by

$$G^{(\alpha,\beta)}(d\cos\theta)=\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)}\sin^{2\alpha+1}\frac{1}{2}\theta\cos^{2\beta+1}\theta\,d\theta.$$

The Jacobi polynomials are the polynomials orthogonal on $[-1, 1]$ under $G^{(\alpha,\beta)}$. We choose the normalisation under which the polynomials take the value 1 at $x=1$; we write the polynomials thus defined as $R_n^{(\alpha,\beta)}$. Then ([19], IV and (7.32.2))

$$\left|R_n^{(\alpha,\beta)}(x)\right| \leq 1 \qquad (x\in[-1, 1]), \tag{1.1}$$

$$\int_{-1}^{1} R_m^{(\alpha,\beta)}(x)\; R_n^{(\alpha,\beta)}(x)\; G^{(\alpha,\beta)}(dx)=\delta_{mn}/w_n^{(\alpha,\beta)} \qquad (n=0,\ 1,\ldots)$$

where

$$w_n^{(\alpha,\beta)}=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha+\beta+2)}\;\frac{(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)\,\Gamma(n+\alpha+1)}{n\,!\;\Gamma(n+\beta+1)}$$

$$=\frac{2\,\Gamma(\beta+1)}{\Gamma(\alpha+1)\,\Gamma(\alpha+\beta+2)}\,n^{2\alpha+1}\,[1+0\,(1/n)]. \tag{1.2}$$

If $f \in L_1(G^{(\alpha,\beta)})$, we write

$$\widehat{f}(n) = \int_{-1}^{1} f(x)\ R_n^{(\alpha,\beta)}(x)\ G^{(\alpha,\beta)}(dx)$$

for the Fourier-Jacobi coefficients of f; we write

$$f(x) \sim \sum_{n=0}^{\infty} \widehat{f}(n)\ w_n^{(\alpha,\beta)}\ R_n^{(\alpha,\beta)}(x)$$

to indicate that the series on the right is the Fourier—Jacobi series of f.

When $\alpha = \beta = \lambda - \frac{1}{2}$ $(\lambda \geq 0)$ we write $R_n^{(\alpha,\beta)}$, $G^{(\alpha,\beta)}$, $w_n^{(\alpha,\beta)}$ as $W_n^{(\lambda)}$, $G^{(\lambda)}$, $w_n^{(\lambda)}$; W_n^{λ} is the nth normalised Gegenbauer or ultraspherical polynomial. When $\alpha = \beta = -\frac{1}{2}$, we have

$$R_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x) = W_n^{(0)}(x) = T_n(x) = \cos(n \cos^{-1} x),$$

$$w_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)} = w_n^{(0)} = 1 \text{ if } n = 0,\ 2 \text{ if } n = 1, 2, \ldots;$$

when $\alpha = \beta = \frac{1}{2}$,

$$R_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(\cos\theta) = W_n^{(1)}(\cos\theta) = \frac{U_n(\cos\theta)}{(n+1)} = \frac{\sin(n+1)\theta}{(n+1)\sin\theta},$$

$$w_n^{\left(\frac{1}{2},\frac{1}{2}\right)} = w_n^{(1)} = (n+1)^2$$

(here T_n, U_n are the Tchebychev polynomials of the first and second kinds). We refer to the cases $\alpha = \beta = -\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}$ as the *cosine*, *sine* and *trigonometric* cases respectively.

Let L be a function slowly varying (at infinity) in Karamata's sense ([16]; [12], [18]). A positive function is *quasi-monotone* if it is of bounded variation on compact subsets of $[0, \infty)$ and if for some (equivalently, for all) $\delta > 0$,

$$\int_0^x t^{\delta} \,|\, df(t) \,| = 0\,(x^{\delta} f(x)) \qquad (x \to \infty).$$

Then (Bojanic-Karamata [11], Th. 1) a slowly varying function L is quasi-monotone if and only if it is of the form $L = f_1/f_2$ with f_i (positive and) non-decreasing with

$$f_i(2x) = 0\,(f_i(x)) \qquad (x \to \infty) \qquad (i = 1, 2).$$

§ 2. Results.

Theorem 1. *Take* $f \in L_1(G^{(\alpha,\beta)})$, $0<\sigma<\alpha+\frac{3}{2}$, *L slowly varying and if* $\alpha+\frac{1}{2}\leq\sigma<\alpha+\frac{3}{2}$ *assume L quasi-monotone. If*

$$\sum_{n=0}^{\infty} |\hat{f}(n)|\, w_n^{(\alpha,\beta)} L(n)/n^{\sigma}<\infty \tag{2.1}$$

then

$$\int_x^{\pi/2} f(\cos\theta)\,\theta^{\sigma} L(1/\theta)\,d\theta/\theta \text{ converges as } x\to 0+. \tag{2.2}$$

Theorem 2. *Take* $f\in L_1(G^{(\alpha,\beta)})$, $\alpha+\frac{1}{2}<\sigma<2\alpha+2$, *L slowly varying with* $L(n+1)-L(n)=0(L(n)/n)$, *and if* $\alpha+\frac{1}{2}<\sigma\leqslant\alpha+\frac{3}{2}$ *assume L quasi-monotone. If*

$$\int_0^{\pi/2} |f(\cos\theta)|\,\theta^{\sigma} L(1/\theta)\,d\theta/\theta<\infty \tag{2.3}$$

and

$$\begin{cases} \displaystyle\int_{\pi/2}^{\pi} \cos^{\sigma-\alpha+\beta}\frac{1}{2}\theta\, L\left(1/\cos\frac{1}{2}\theta\right)|f(\cos\theta)|\,d\theta<\infty & \left(\alpha+\frac{1}{2}<\sigma<\alpha+\beta+1\right)\\ \displaystyle\int_{\pi/2}^{\pi} \cos^{2\beta+1}\frac{1}{2}\theta\, L^{*}\left(1/\cos\frac{1}{2}\theta\right)|f(\cos\theta)|\,d\theta<\infty & (\sigma=\alpha+\beta+1)\end{cases} \tag{2.4}$$

(here $L^{*}(x)=\int_1^x L(u)\,du/u$), *then*

$$\sum_{n=0}^{\infty} \hat{f}(n)\, w_n^{(\alpha,\beta)} L(n)/n^{\sigma} \text{ converges.} \tag{2.5}$$

Note that in Theorems 1,2 our main hypotheses (2.1) and (2.3) (together with the antipole condition (2.4)) involve absolute convergence or summability, while our conclusions (2.2), (2.5) involve conditional convergence or integrability in the Cauchy limit sense. This is also true of the results of Robertson [17], who restricts himself to the trigonometric cases but treats functions rather more general than the regularly varying functions $L(n)/n^{\sigma}$ used here.

On the other hand, in the trigonometric cases with $L(.)\equiv 1$, Heywood ([14], [15]) obtains results which improve the corresponding specialisations of Theorems 1,2 by weakening the summability required in the hypotheses from absolute to conditional (or Cauchy). Heywood's methods require the Wiener

Tauberian theory; the connection of Tauberian methods and of Heywood's work with the results above will be discussed further in § 5.

Note also that if we restrict attention to functions f with $\hat{f}(n)\geq 0$ for all large enough n, (2.1) and (2.5) coincide and Theorems 1 and 2 become more symmetrical. In the trigonometric cases, and with $L(.)=1$, Boas [9] proves that with $\hat{f}(n)\geq 0$ convergence of the series is equivalent to the Cauchy-integrability condition (2.2) at the origin (which in turn is equivalent to a Cauchy-integrability condition at each other point of $[0, \pi)$). See also Ganser [13] for the ultraspherical case $\alpha=\beta$ with $L(.)=1$.

Under more stringent conditions on $\hat{f}(n)$, more is true. With $\hat{f}(n)\downarrow 0$, convergence of the series is equivalent to absolute integrability. Results for the trigonometric cases with L as above were given by Aljančić, Bojanić and Tomić [3]. Monotonicity of $\hat{f}(n)$ can be weakened to $\hat{f}(n+2)-\hat{f}(n)\geq 0$ (Ganser [13], Boas [10]) or to quasi-monotonicity (in Szász's sense; Yong [20]). Analogues with monotonicity conditions on $f(\cos\theta)$ rather than $\hat{f}(n)$ were obtained by Adamović [1], [2]; cf. Boas [10].

§ 3. Proof of theorem 1.

Take $r\in(0,1)$, and write

$$f_r(\cos\theta)=\sum_{n=0}^{\infty} r^n\hat{f}(n)\, w_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta)$$

for the Abel-sum of the Fourier-Jacobi series of f. Then (see, e.g. Bavinck [5], Th. 2.4) $f_r\to f$ in L_1-norm as $r\to 1-$. So $f_r g\to fg$ in L_1-norm for bounded g. Thus for $x\in\left(0, \frac{1}{2}\pi\right)$ we have

$$\begin{aligned}&\int_x^{\frac{1}{2}\pi}\theta^\sigma L(1/\theta)\,d\theta/\theta \sum_{n=0}^{\infty} r^n\hat{f}(n)\, w_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta)\\ &\to\int_x^{\frac{1}{2}\pi}\theta^\sigma L(1/\theta) f(\cos\theta)\,d\theta/\theta \qquad (r\to 1-).\end{aligned}\tag{3.1}$$

By (1.1), (1.2) and dominated convergence, the left-hand side is

$$\sum_{n=0}^{\infty} r^n\hat{f}(n)\, w_n^{(\alpha,\beta)}\int_x^{\frac{1}{2}\pi}\theta^\sigma L(1/\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\,d\theta/\theta.\tag{3.2}$$

Since $\sigma>0$, (1.1) shows that the integral in (3.2) converges as $x\to 0+$. We next show that this integral is $0(L(n)/n^\sigma)$ uniformly in x:

Lemma 1. *Take* $0<\sigma<\alpha+\frac{3}{2}$, *and if* $\alpha+\frac{1}{2}\leqslant\sigma<\alpha+\frac{3}{2}$ *assume* L *quasi-mono tone. Then*

$$\int_x^{\frac{1}{2}\pi} \theta^\sigma L(1/\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, d\theta/\theta = 0\,(L(n)/n^\sigma),$$

uniformly for $x\in\left(0,\ \frac{1}{2}\pi\right)$.

Proof. Choose $\delta>0$ arbitrarily small. If $x\geq\delta/n$, (1.1) gives

$$\begin{aligned}\left|\int_x^{\frac{1}{2}\pi} \theta^\sigma L(1/\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, d\theta/\theta\right| &\leq \int_{\delta/n}^{\frac{1}{2}\pi} \theta^\sigma L(1/\theta)\, d\theta/\theta \\ &= \int_{2/\pi}^{n/\delta} t^{-\sigma} L(t)\, dt/t \\ &\sim cL(n)/n^\sigma \qquad (n\to\infty).\end{aligned}$$

This also deals with the contribution $\int_{\delta/n}^{\frac{1}{2}\pi}$ if $x<\delta/n$. If $0<x<\delta/n$, we use the asymptotic formula

$$R_n^{(\alpha,\beta)}(\cos\theta)=\frac{\Gamma(\alpha+1)}{\left[n+\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\right]^\alpha}\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{J_\alpha\left(n\theta+\frac{1}{2}\theta(\alpha+\beta+1)\right)}{\sin^\alpha\frac{1}{2}\theta\cos^\beta\frac{1}{2}\theta} \tag{3.3}$$
$$+0\,(\theta^2)$$

([19], 197, Th. 8.21.12). The contribution of the error term is

$$O\left(\int_0^x \theta^2\cdot\theta^\sigma L(1/\theta)\, d\theta/\theta\right)=O\left(\int_0^{\delta/n} \theta^{\sigma+2} L(1/\theta)\, d\theta/\theta\right)=O\,(L(n)/n^{\sigma+2}).$$

The contribution of the main term is asymptotic to

$$\frac{c}{n^\alpha}\int_x^{\frac{1}{2}\pi}\frac{J_\alpha\left(n\theta+\frac{1}{2}\theta(\alpha+\beta+1)\right)}{\sin^\alpha\frac{1}{2}\theta\cos^\beta\frac{1}{2}\theta}\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\theta^\sigma L(1/\theta)\, d\theta/\theta$$

$$= cn^{-\alpha} \int_{x}^{\frac{1}{2}\pi} \theta^{-\alpha} J_\alpha\left(n\theta + \frac{1}{2}\theta(\alpha+\beta+1)\right) \cdot \theta^\sigma L_1(1/\theta)\, d\theta/\theta \qquad (\text{with } L_1 \sim L)$$

$$= cn^{-\sigma} \int_{nx}^{\frac{1}{2}n\pi} u^{-\alpha} J_\alpha\left(u + \frac{u}{2n}(\alpha+\beta+1)\right) u^\sigma L_1(n/u)\, du/u.$$

Recalling that $0 < nx < \delta$ and δ is arbitrarily small, this is easily seen to reduce to

$$cn^{-\sigma} \int_0^\infty u^{-\alpha} J_\alpha(u) \cdot u^\sigma L_1(n/u)\, du/u \tag{3.4}$$

plus terms of smaller order. If $0 < \sigma < \alpha + \frac{1}{2}$ the integral above is absolutely convergent, and is asymptotic to

$$cL(n)\, n^{-\sigma} \int_0^\infty u^{\sigma-\alpha} J_\alpha(u)\, du/u \tag{3.5}$$

as $n \to \infty$ (cf. Bojanić-Karamata [11], Th. 5, [8]). If $\alpha + \frac{1}{2} \leqslant \sigma < \alpha + \frac{3}{2}$ and L is quasi-monotone the same estimate follows by another result of Bojanić and Karamata ([11], Th. 6; [8]); here the integral in (3.5) is only conditionally convergent and that in (3.4) is taken in the Cauchy-limit sense.

By the lemma and hypothesis (2.1), we can use dominated convergence to let $r \to 1-$ in (3.2), obtaining

$$\sum_{n=0}^\infty \widehat{f}(n)\, w_n^{(\alpha,\beta)} \int_x^{\frac{1}{2}\pi} \tilde{\theta}^\sigma L(1/\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, d\theta/\theta = \int_x^{\frac{1}{2}\pi} \theta^\sigma L(1/\theta) f(\cos\theta)\, d\theta/\theta.$$

By the lemma and (1.1) again, we can now use dominated convergence to let $x \to 0+$; this yields (2.2) and completes the proof.

§ 4. Proof theorem 2.

Our method of proof here is analogous to that used by Ganser [13] for the ultraspherical case, extended to the Jacobi case by the techniques of [6]. We have

$$\sum_{n=0}^N \widehat{f}(n)\, w_n^{(\alpha,\beta)} L(n)/n^\sigma = \int_0^\pi f(\cos\theta) \left\{\sum_{n=0}^N \frac{L(n)}{n^\sigma} w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)\right\}$$

$$\cdot\, G^{(\alpha,\beta)}(d\cos\theta). \tag{4.1}$$

We obtain bounds on the expression in braces, uniform in N, on $\left(0, \frac{1}{2}\pi\right]$ and $\left[\frac{1}{2}\pi, \pi\right)$, from which (2.5) will follow from (2.3) and (2.4).

Lemma 2. If $\alpha+\frac{1}{2}<\sigma<2\alpha+2$,

$$\sum_{n=0}^{N} \frac{L(n)}{n^\sigma} w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) = 0\,(\theta^\sigma L(1/\theta)/\theta^{2\alpha+2}) \qquad \left(\theta\in\left(0, \frac{1}{2}\pi\right]\right)$$

uniformly in N.

Proof. If $N<1/\theta$,

$$\left|\sum_{n\le N} \frac{L(n)}{n^\sigma} w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)\right| \le c \sum_{n\le 1/\theta} \frac{L(n)}{n^\sigma} n^{2\alpha+1} = 0\,(\theta^\sigma L(1/\theta)/\theta^{2\alpha+2}).$$

In what follows, we may thus suppose $N>1/\theta$. Write

$$U_n(0) = \sum_{k=0}^{n} w_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(\cos\theta).$$

By the Christoffel—Darboux identity Jacobi polynomials ([19], 71; [6])

$$U_n(\theta) = \frac{(\alpha+\beta+2)}{(2n+\alpha+\beta+2)} w_n^{(\alpha+1,\beta)} R_n^{(\alpha+1,\beta)}(\cos\theta).$$

Using Szego's asymptotic formula for the Jacobi polynomials ([19], Th. 8.21.13) this gives

$$U_n(\theta) = 0\,(n^{\alpha+\frac{1}{2}}/\theta^{\alpha+\frac{1}{2}}(\pi-\theta)^{\beta+\frac{1}{2}}) \qquad (\delta/n\le\theta\le\pi-\delta/n).$$

Suppose for the moment that $1/\theta$ is an integer. Summing by parts,

$$\sum_{1/\theta<n\le N} \frac{L(n)}{n^\sigma} w_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) = -\sum_{1/\theta<n<N} \Delta\left(\frac{L(n)}{n^\sigma}\right) U_n(\theta)$$
$$-U_{1/\theta}(\theta)\,\theta^\sigma L(1/\theta) + U_N(\theta)\,L(N)/N^\sigma.$$

By assumption, $\Delta L(n) = 0\,(L(n)/n)$, and so $\Delta(L(n)/n^\sigma) = 0\,(L(n)/n^{\sigma+1})$. For $\theta\in\left(0, \frac{1}{2}\pi\right]$, the sum on the right is thus

$$0\left(\theta^{-\left(\alpha+\frac{3}{2}\right)} \sum_{1/\theta<n<N} L(n)/n^{\sigma-\left(\alpha+\frac{1}{2}\right)+1}\right) = 0\,(\theta^\sigma L(1/\theta)/\theta^{2\alpha+2})$$

since $\sigma>\alpha+\frac{1}{2}$. The second term on the right is also $0\,(\theta^\sigma L(1/\theta)/\theta^{2\alpha+2})$ (and so is the term omitted above if $1/\theta$ is not an integer). Since $\sigma>\alpha+\frac{1}{2}$, $L(n)n^{\alpha+\frac{1}{2}}/n^\sigma$ is regularly varying with negative index, and is thus asymptotically decreasing. So for $N>1/\theta$, $L(N)\,N^{\alpha+\frac{1}{2}}/N^{\sigma}=0\,(\theta^{\sigma}L(1/\theta)/\theta^{\alpha+\frac{1}{2}})$. Combining, the lemma follows.

Lemma 3. *For* $\theta\in\left[\frac{1}{2}\pi,\ \pi\right)$, *write* $\phi=\pi-\theta$. *Then*

$$\sum_{n=0}^{N}\frac{L(n)}{n^{\sigma}}\,w_n^{(\alpha,\beta)}\,R_n^{(\alpha,\beta)}(\cos\theta)=\begin{cases}0\,(1) & \text{if } \sigma>\alpha+\beta+1\\ 0\,(L^*(1/\phi)) & \text{if } \sigma=\alpha+\beta+1\\ 0\,(\phi^{\sigma}L(1/\phi)/\phi^{\alpha+\beta+1}) & \text{if } \sigma<\alpha+\beta+1\end{cases}$$

uniformly in N $\left(\textit{here } L^*(x)=\int_1^x L(t)\,dt/t\right)$.

Proof. In terms of the un-normalised Jacobi polynomials P_n ([19], IV)

$$R_n^{(\alpha+1,\beta)}(\cos\theta)=\frac{n!\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}\,P_n^{(\alpha+1,\beta)}(\cos\theta)$$

$$=\frac{(-)^n\,n!\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}\,P_n^{(\beta,\alpha+1)}(\cos\phi)$$

(Szegö [19], (4.1.3))

$$=(-)^n\,\frac{\Gamma(\alpha+2)}{\Gamma(\beta+1)}\,\frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+2)}\,R_n^{(\beta,\alpha+1)}(\cos\phi).$$

But

$$1-R_n^{(\alpha,\beta)}(\cos\theta)=0\,\Big[(1-\cos\theta)\sup_{[-1,1]}\big|DR_n^{(\alpha,\beta)}(.)\big|\Big]=0\,(\theta^2 n^2)$$

(Szegö [19], (7.32.10)), and so if $0<\phi<\delta/n$

$$R_n^{(\alpha+1,\beta)}(\cos\theta)=0\,(1/n^{\alpha+1-\beta}),$$

$$U_n(\theta)=0\,(n^{-1}\cdot n^{2\alpha+3}/n^{\alpha+1-\beta})=0\,(n^{\alpha+\beta+1}).$$

As above, we suppose for the moment that $1/\phi$ is not an integer and sum by parts: if $N\leq 1/\phi$,

$$\sum_{n\leq N}\frac{L(n)}{n^{\sigma}}\,w_n^{(\alpha,\beta)}\,R_n^{(\alpha,\beta)}(\cos\theta)=-\sum_{n<N}\Delta\left(\frac{L(n)}{n^{\sigma}}\right)U_n(\theta)+L(N)\,N^{-\sigma}\,U_N(\theta)$$

$$=-\sum_{n<N}0\,(L(n)/n^{\sigma+1})\cdot 0\,(n^{\alpha+\beta+1})+0\,(L(N)\cdot N^{-\sigma}\cdot N^{\alpha+\beta+1}).$$

The term omitted if $1/\phi$ is not an integer is $0\,(L(1/\phi)\,\phi^{\sigma}/\phi^{\alpha+\beta+1})$. If $\sigma>\alpha+\beta+1$, the right-hand iside is $0\,(1)$. If $\sigma<\alpha+\beta+1$, both terms on the right are $0\,(N^{-\sigma}\cdot L(N)\cdot N^{\alpha+\beta+1})=0\,(\phi^{\sigma}L(1/\phi)/\phi^{\alpha+\beta+1})$. If $\sigma=\alpha+\beta+1$, the first term on the right is

$$0\,(\sum_{n<N}L(n)/n)=0\,(L^*(N))=0\,(L^*(1/\phi)).$$

Combining, the lemma follows if $N\leq 1/\phi$. This also deals with the sum over $0\leq n\leq 1/\phi$ if $N>1/\phi$.

On the other hand, if $N>1/\phi$

$$\sum_{1/\phi<n\leq N}\frac{L(n)}{n^\sigma}\,w_n^{(\alpha,\beta)}\,R_n^{(\alpha,\beta)}(\cos\theta)$$

$$=-\sum_{1/\phi<n<N}\Delta\left(\frac{L(n)}{n^\sigma}\right)U_n(\theta)-\phi^\sigma L(1/\phi)\,U_{1/\phi}(\theta)+N^{-\sigma}L(N)\,U_N(\theta).$$

We use Szego's asymptotic formula as above. Since θ is bounded away from zero, this is

$$\sum_{1/\phi<n<N} 0\left(\frac{L(n)}{n^{\sigma+1}}\frac{n^{\alpha+\frac{1}{2}}}{\phi^{\beta+\frac{1}{2}}}\right)+0\left(\phi^\sigma L(1/\phi)/\phi^{\alpha+\beta+1}\right)+0\left(\frac{L(n)}{n^\sigma}\frac{N^{\alpha+\frac{1}{2}}}{\phi^{\beta+\frac{1}{2}}}\right).$$

As $\sigma>\alpha+\dfrac{1}{2}$, the sum on the right is

$$0\left(\phi^{-\left(\beta+\frac{1}{2}\right)}\sum_{n>1/\phi}L(n)\cdot n^{\alpha+\frac{1}{2}}/n^{\sigma+1}\right)=0\left(\phi^\sigma L(1/\phi)/\phi^{\alpha+\beta+1}\right)$$

and similarly, so is the third term as $L(n)\cdot n^{\alpha+\frac{1}{2}}/n^\sigma$ is asymptotically decreasing. Thus

$$\sum_{1/\phi<n<N}\frac{L(n)}{n^\sigma}\,w_n^{(\alpha,\beta)}\,R_n^{(\alpha,\beta)}(\cos\theta)=0\left(\phi^\sigma L(1/\phi)/\phi^{\alpha+\beta+1}\right).$$

Combining, the lemma follows.

From Lemmas 2 and 3, $\sum_{n=0}^{N} n^{-\sigma}L(n)\,w_n^{(\alpha,\beta)}\,R_n^{(\alpha,\beta)}(\cos\theta)$ is bounded on compact subsets of $(0,\pi)$ uniformly in N, and satisfies uniform 0-conditions at 0 and π corresponding to the integrability hypotheses (2.3), (2.4) on f.

Write $\rho=2\alpha+2-\sigma$; then $0<\rho<\alpha+\dfrac{3}{2}$ since $\alpha+\dfrac{1}{2}<\sigma<2\alpha+2$. By the results of [6] with σ replaced by ρ, $\sum_{n=0}^{\infty} n^{-\sigma}L(n)\,w_n^{(\alpha,\beta)}\,R_n^{(\alpha,\beta)}(\cos\theta)$ converges on $(0,\pi)$ $\left(\text{recall } L \text{ is quasi-monotone if } \alpha+\dfrac{1}{2}<\sigma\leq\alpha+\dfrac{3}{2}, \text{ that is, } \alpha+\dfrac{1}{2}\leq\rho<\alpha+\dfrac{3}{2}\right)$. Hence both $\sum_{n=0}^{\infty} n^{-\sigma}L(n)\,w_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta)$ and $\sum_{n>N} n^{-\sigma}L(n)\,w_n^{(\alpha,\beta)}$ $R_n^{(\alpha,\beta)}(\cos\theta)$ satisfy the uniform 0-conditions of Lemmas 2 and 3.

We now return to (4.1), and write its right-hand side as

$$\int_0^\pi\sum_{n\leq N}=\int_0^{1/N}\sum_{n\leq N}+\int_{1/N}^{\pi}\sum_{n=0}^{\infty}-\int_{1/N}^{\pi}\sum_{n>N}$$

in an obvious notation. Using dominated convergence, the uniform 0-estimates above and (2.3), (2.4), the second term on the right converges as $N \to \infty$ while the first and third terms tend to zero. Thus the left-hand side of (4.1) converges as $N \to \infty$, which proves Theorem 2 as required.

§ 5. Complements.

1. Theorems 1,2 link the behaviour of $f(\cos\theta)$ at the origin with that of $\hat{f}(n)$ at infinity. Other such links are known: for example, in [6] the relationship between regular variation of $f(\cos\theta)$ and $\hat{f}(n)$ is thoroughly explored (cf. Aljančić-Bojanić-Tomić [4]). In [6], the implication from $\hat{f}(n)$ to $f(\cos\theta)$ (as in Theorem 1) is Abelian in nature and that from $f(\cos\theta)$ to $\hat{f}(n)$ (as in Theorem 2) Tauberian. We regard Theorems 1,2 here as Abelian and Tauberian, by analogy and for reasons to be discussed further below, although it is significant that Tauberian methods play no explicit role in the present paper.

2. For the reasons for the different roles played in Theorems 1 and 2 by the strips $0<\sigma<\alpha+\frac{1}{2}$, $\alpha+\frac{1}{2}<\sigma<\alpha+\frac{3}{2}$, $\alpha+\frac{3}{2}<\sigma<2\alpha+2$ we refer to [6] (where they are called the strips of absolute convergence, conditional convergence and Abel summability). We point out that Theorems 1 and 2 become more symmetrical when attention is restricted to the strip of conditional convergence.

3. The results obtained here and in [6] have analogues for Hankel transforms, essentially because of the occurrence of the Bessel functions in the asymptotic formula (3.3) (cf. [6] for details). The question then arises of extending these analogues to integral transforms with more general kernels. This is done at length in [7] and [8], using the Wiener Tauberian theory. It is interesting to note that, whereas slowly varying functions L play a natural role throughout [6], [8] and the present paper, in [7] we must confine ourselves to the case $L(.)=1$ (cf. [7] for counter-examples). This essential restriction is reflected here in the lack of a Tauberian result (in Theorem 2) when $0<\sigma<\alpha+\frac{1}{2}$.

4. In the trigonometric cases $\alpha=\beta=\pm\frac{1}{2}$ the results mentioned above for Hankel tran forms specialise to Fourier cosine and sine transforms. Corresponding results for cosine and sine series (with $L(.)=1$) were obtained by Heywood [14], [15]. The Wiener Tauberian methods used in [14], [15] and [7] provide results in which Cauchy integrability (or conditional convergence) appears in both hypothesis and conclusion. Thus Heywood's results improve the cases $\alpha=\beta=\pm\frac{1}{2}$, $L(.)=1$ of ours, not only by weakening the hypotheses from absolute to Cauchy (or conditional) summability, but also by filling the gaps in the parameter-ranges $\left(\alpha+\frac{3}{2}<\sigma<2\alpha+2\right.$ in Th. 1, $0<\sigma<\alpha+$

$\left.\frac{1}{2} \text{ in Th. } 2\right)$. The question arises as to whether the case $L(.)=1$ of our results with general α and β is capable of similar improvement, but this we leave open.

5. Other types of integrability results relating $f(\cos\theta)$ to $\hat{f}(n)$ are also possible. This is sufficiently exemplified by the trigonometric cases, which (as the discussion above indicates) are particularly interesting. For many other formulations in these cases, and a wealth of additional information, we refer to Boas' monograph [10].

6. We have made two restrictions on the slowly varying functions L, namely quasi-monotonicity and $\Delta L(n)=0(L(n)/n)$. For the first, we refer to Bojanic and Karamata [11], where slowly varying quasi-monotone functions are characterised and their great use in asymptotic estimates such as in Lemma 1 (and in the convergence properties of series such as Fourier) is explained in detail. For the second, we recall Karamata's characterisation of slowly varying functions as those with

$$L(x)=c(x)\exp\left(\int_1^x \varepsilon(u)\,du/u\right)$$

with $c(x)\to c\in(0,\infty)$, $\varepsilon(x)\to 0$ as $x\to\infty$. This representation is not unique, and the smoothness properties of $L(x)$ are limited by those obtainable for $c(x)$. For those $L(.)$ for which $c(.)$ may be replaced by c (called normalised varying functions), one has

$$kL'(x)/L(x)\to 0 \qquad (x\to\infty).$$

The restriction $\Delta L(n)=0(L(n)/n)$ is somewhat similar but very much weaker. It provides a mild degree of control over the oscillations of $c(.)$, and is dictated by our use of the Christoffel-Darboux identity, without which the orthogonality property cannot be properly exploited.

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