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BOUNDED FUNCTIONS WITH NO SPECTRAL GAPS*

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Let $\lambda=(\lambda_n)_{-\infty}^{+\infty}$ be a monotonically increasing sequence of real numbers, and let E be the union of all the intervals $(\lambda_{2n}, \lambda_{2n+1})$. Several conditions are given which are sufficient for the characteristic function of E to have no spectral gaps.

1. Notation, Definitions, and Facts

Let $f\in L^\infty(R)$. The Fourier-Carleman functions

$$F^+(f, z)=\int_{-\infty}^{0} f(t)\, e^{-izt}\, dt$$

and

$$F^-(f, z)=-\int_{0}^{\infty} f(t)\, e^{-izt}\, dt$$

are holomorphic in the upper and lower half-planes, respectively. Any open interval on the real line across which $F^+(f, z)$ and $F^-(f, z)$ continue analytically to each other is salled a spectral gap for the function f. The spectrum of f, denoted by $\sigma(f)$, is the set of all points on the real line which do not belong to any spectral gap of f.

The following facts are well known [6, Chapter VI]: if $\sigma(f)=\varnothing$, f is zero; if $\sigma(f)=\{0\}$, f is a constant; if f is periodic, $\sigma(f)$ is discrete; f has the open interval I as a spectral gap if and only if $f*g=0$ for every $g\in L^1(R)$ such that the support of $\hat{g}$ (the Fourier transform of g) is a compact subset of I.

If g and h are locally integrable functions on R, we set

$$D_S(g, h)=\sup_{x\in R}\int_{x}^{x+1} |g(t)-h(t)|\, dt.$$

The closure of the space of trigonometric polynomials with respect to the distance D_S is the collection of Stepanoff almost periodic — shortly: SAP — functions.

We shall need the following fact [1, p. 104]: if f is a SAP function, the expression

$$\frac{1}{2T}\int_{-T}^{T} f(x)\, e^{-i\lambda x}\, dx$$

tends to a limit $a(f, \lambda) = a(\lambda)$ for every real number λ as $T \to \infty$. The values of λ for which $a(f, \lambda) \neq 0$ are called the Fourier exponents of f.

Let $\lambda = (\lambda_n)_{-\infty}^{+\infty}$ be a monotonically increasing sequence of real numbers, and let E be the union of all the intervals $(\lambda_{2n}, \lambda_{2n+1})$. A set E obtained in this manner we will denote by $E = E(\lambda)$.

A function which takes only finitely many values and whose discontinuities from a discrete set is called a step function.

2. Introduction

There are two extreme cases in which a non-constant bounded real-valued function g has no spectral gaps:

(1) if one of the two Fourier-Carleman functions $F^+(g, z)$ and $F^-(g, z)$ has the real line as its natural boundary;

(2) if one of $F^+(g, z)$ and $F^-(g, z)$ is analytically continuable across the whole real axis, or across the whole real axis except at the origin.

The statement is obvious for case (1). In case (2), if g had a spectral gap, it would follow by analytic continuation that $\sigma(g) = \varnothing$ or $\{0\}$, so that g would be a constant, contrary to the assumption.

Computing the Fourier-Carleman functions for χ_E, where $E = E(\lambda)$, we easily deduce from (1) and (2) that χ_E will have no spectral gap if one of the two Dirichlet series

$$f^+(\lambda, z) = \sum_{n=-\infty}^{-1} (-1)^{n+1} e^{-i\lambda_n z}, \quad \operatorname{Im} z > 0,$$

and

$$f^-(\lambda, z) = \sum_{n=0}^{\infty} (-1)^n e^{-i\lambda_n z}, \quad \operatorname{Im} z < 0,$$

is analytically contınuable across the whole real axis, or if one of the two series has the real axis as its natural boundary. From this remark we obtan examples of sequences $\lambda = (\lambda_n)_{-\infty}^{+\infty}$ such that $\lambda_{E(\lambda)}$ has no spectral gaps.

Example 1. $\lambda_{n-1} = \log n$, $n \geq N$, N a positive integer.

If $\lambda_{n-1} = \log n$, $n \geq 1$, then

$$f^-(\lambda, z) = \sum_{n=0}^{\infty} (-1)^n (n+1)^{-iz} = (1-2^{1-iz}) \zeta(iz)$$

is an entire function, ζ representing the Riemann zeta-function.

If $\lambda_{n-1} = \log n$, $n \geq N$, $f^-(\lambda, z)$ differs from $(1-2^{1-iz})\zeta(iz)$ by a finite linear combination of functions of from e^{-iaz}, so that $f^-(\lambda, z)$ is again an entire function.

Example 2. $\lambda_n = n^\alpha$, $n \geq N$, N a positive integer, $0 < \alpha < 1$.

It is sufficient to consider the case $\lambda_n = n^\alpha$ for $n \geq 0$. In that case, it is a result of Hardy [5] that

$$f^-(\lambda, z) = \sum_{n=0}^{\infty} (-1)^n e^{-in^\alpha z}$$

is an entier function.

Example 3. $\lim\limits_{n\to\infty} \dfrac{\lambda_n}{n} = \infty$.

In this case it follows from Fabry's gap theorem [9] that $f^-(\lambda, z)$ either has the real axis as natural boundary or is analytically continuable across the whole real axis.

The questions we are considering are special cases of the general problem: for which measurable subsets E of the real line does the characteristic function of E have no spectral gaps? The analogous general problem, which arises when the real line is replaced with the group Z of integers, is easy to solve. Namely, if E is a subset of Z, the characteristic function of E has a spectral gap, i.e., the series

$$\sum_{\substack{n\in E\\ n\geq 0}} z^n \quad \text{and} \quad \sum_{\substack{n\in E\\ n<0}} z^n$$

continue analytically across some arc of the unit circle into each other, if and only if the set E is periodic. This follows from the classical theorem of Szegö on power series the coefficients of which take only finitely many distinct values [4, pp. 324—327].

The situation is completely different on the real line: there are non-periodic subsets of the real line whose characteristic functions have spectral gaps. A simple example of such a set, due to Logan [8], is $\{x \mid \cos(x + 2\arctan x) > 0\}$. Logan and Shapiro have obtained complete characterizations of bounded functions, characteristic functions of measurable sets, and characteristic functions of sets of form $E(\lambda)$ which have the interval $(-a, a)$ as a spectrrl gap (Theorems 7.6.3, 7.6.5.6, and 7.6.6.7 in [8]).

3. Theorems and open questions

If a bounded measurable function s has for spectrum the whole real line, then so does any translate of s. Our first theorem is a generalization of this elementary fact.

Theorem 1. *Let s and q be bounded measurable functinns on the real line and let $\sigma(s)$ be the whole real line. If there exists a sequence A_n of measurable sets such that*

i) $\chi_{A_n} \to 1$ *almost everywhere as $n \to \infty$ and*

ii) *for every n there is a translate q_n of q such that $q_n(t) = s(t)$ for t in A_n, then $\sigma(q)$ is the whole real line.*

Applying this theorem to the case where $q = \chi_{E(\lambda)}$, $A_n = (\lambda_n - \lambda_{n+1}, \lambda_{n+2} - \lambda_{n+1})$, $q_n(t) = q(t + \lambda_{n+1})$, and where s is the characteristic function of either the positive or negative half-line, we obtain the following corollary.

Corollary. If $\lambda = (\lambda_n)_{-\infty}^{+\infty}$ is an increasing sequence of real numbers and if the sequence $m_n = \min\{\lambda_{n+1} - \lambda_n, \lambda_{n+2} - \lambda_{n+1}\}$ is unbounded, then the characteristic function of the set $E(\lambda)$ has no spectral gaps.

The unboundedness of the sequence in the corollary imposes a thinness on the exponents in two Dirichlet series and the conslusion concerns the analytic continuation of those series. Thus the corollary gains independent interest because it is near the classical group of gap-type theorems, whose purpose is to make conclusions about analytic continuation of functions defined by series on the basis of the sizes of gaps between exponents.

It should be observed that neither of the two conditions — the one in the corollary and the one in Example 3 — implies the other. We do not know whether the condition in the corollary can be replaced by the weaker condition that the sequence $\lambda_{n+1} - \lambda_n$ is unbounded.

The set $E = \{x \mid \cos(x + 2\arctan x) > 0\}$, the characteristic function of which has a spectral gap around zero, is the union of the intervals $(\lambda_{2n}, \lambda_{2n+1})$, where λ_n is the unique root of the equation $x + 2\arctan x = \pi n - \dfrac{\pi}{2}$, so that $\lambda_n = \pi n - \dfrac{3\pi}{2} + 0(|n|^{-1})$. Thus, in this example, the sequence $(\lambda_n)_{-\infty}^{-\infty}$ is, in a sence, close to the periodic sequence $\left(\pi n - \dfrac{3\pi}{2}\right)_{-\infty}^{+\infty}$. One might expect that, whenever λ is close to a periodic sequence, the characteristic function of $E(\lambda)$ will have a spectrall gap, an expectation which might be strengthened by the corollary, where λ was far from being periodic and where $\chi_{E(\eta)}$ had no spectral gaps. Actually, quite the opposite is true. If the periodic sequence $p = (p_n)_{-\infty}^{+\infty}$ is perturbed, namely if it is replaced by the slightly nonperiodic sequence $\lambda = (\lambda_n)_{-\infty}^{+\infty}$ such that

$$|\lambda_n - p_n| = 0(e^{-\beta n}) \quad \text{as} \quad n \to +\infty, \ \beta > 0, \tag{3}$$

then the spectrum is drastically changed: $\sigma(\chi_{E(p)})$ was discrete, but $\sigma(\chi_{E(\lambda)})$ is the whole real line. More generally, we have the following theorem.

Theorem 2. *Let P be a measurable periodic set, E a measurable non--periodic set of positive measure, and S the symmetric difference of E and P. If there exists an $\alpha>0$ such that*

$$\int_{S\cap(0,\infty)} e^{\alpha t}\,dt<\infty, \tag{4}$$

then χ_E has no spectral gaps.

We observe that, if condition (3) is satisfied, then (4) is al o satisfied with S being the symmetric difference of $E(p)$ and $E(\lambda)$. For, since p is a periodic sequence, $p_{n+1}-p_n\geq\gamma>0$, and, since also $\lambda_n-p_n\to 0$, we have $p_{n-1}<\lambda<p_{n+1}$ for n large enough. Thus, for a large L, $S\cap(L,\infty)$ is contained in a union of intervals I_n, $n\geq N$, the endpoints of I_n being λ_n and p_n. Since p is periodic, we can find $M>0$ such that $p_n\leq Mn$ and $\lambda_n\leq Mn$ for $n\geq N$, and so

$$\int_{I_n} e^{\alpha t}\,dt=|\lambda_n-p_n|\,e^{\alpha Mn}\leq Ce^{(\alpha M-\beta)n},$$

so that (4) is satisfied with suitably small $\alpha>0$.

We do not know whether Theorem 2 remains valid if (4) is replaced by by the weaker condition $m(S)<\infty$.

Let $p_1=1$, $p_2=\pi$, and $p_3=\dfrac{\pi}{\pi+1}$. Let E_ν, $\nu=1,2,3$, be the union of the intervals $(2np_\nu,\ (2n+1)p_\nu)$, $n\in Z$. Then, if $D=E_1\cap E_2\cap E_3$, χ_D is the product of the characteristic functions of the $E_\nu's$, and is a non-periodic SAP function. If the Fourier series of the $\chi_{E_\nu}'s$ are computed, if these series are formally multiplied, and if all terms with a common exponent are grouped, it is found that the only exponents whose associated coefficients do not vanish (those coefficients which do vanish present themselves as series which are summable in closed from) are contained in a union of three arithmetic progressions, so that, by Lemma 2 which appears below, χ_D has spectral gaps. The function in this example, communicated in a letter from Bernard Ploeger, University of Dayton, Dayton, Ohio, is close to being periodic, but in an arithmetically different way than were the characteristic functions of the perturbed sets described prior to Theorem 2. It is not true that every non-periodic $SAP\,\chi_{E(\lambda)}$ has spectral gaps, as we show in our last heorem.

Theorem. 3. *Let each of the periodic sets E_ν, $1\leq\nu\leq k$, $k\geq 2$, be a union of disjoint intervals whose endpoint form a discrete set, and let D and U denote the intersection and union, respectively, of all the $E_\nu's$. Let p_ν be the period of E_ν. If the numbers p_ν^{-1}, $1\leq\nu\leq k$, are rationally indepedent, then $\sigma(\chi_D)$ and $\sigma(\chi_U)$ are both the whole real line.*

Ploeger's examples show thaȷ the rational independence of the inverses of the periods cannot be weakend to pairwise rational independence. The characteristic functions in Theorem 3 are also non-eriodic SAP functions, making it desirable to have a characterization, probably arithmetic, of the non-periodic $SAP\,\chi_{E(\lambda)}'s$ which have spectral gaps. We do not have such a characterization.

We shall actually prove a slightly more general result than Theorem 3.

Theorem 3'. *Let* f_ν, $1 \leq \nu \leq k$, $k \geq 2$, *be real-valued nonconstant periodic step functions with periods* p_ν, $1 \leq \nu \leq k$, *and let* $f = f_1, \ldots, f_k$. *If the numbers* p_ν^{-1}, $1 \leq \nu \leq k$, *are rationally independent, then* $\sigma(f)$ *is the whole real line.*

Theorem 3 follows from Theorem 3' since $\chi_D = \prod_{\nu=1}^{k} \chi_{E_\nu}$ and $\chi_U = 1 - \prod_{\nu=1}^{k} \chi_{C(E_\nu)}$.

We conjecture that any function in the algebra generated by periodic step functions is either a finite sum of periodic functions or has spectrum the whole real line. Theorem 3' is a special case of this conjecture whose proof apears to rest on deep results from outside Harmonic Analysis. This proof is based on the following three lemmas.

Lemma 1. *Let* f_ν, $1 \leq \nu \leq k$, $k \geq 2$, *be bounded measurable periodic functions of bounded variation with periods* p_ν. *If the numbers* p_ν^{-1}, $1 \leq \nu \leq k$, *are rationally independent, then the closure of the set* $\sigma(f_1) + \sigma(f_1) + \cdots + \sigma(f_k)$ *is contained in* $\sigma(f_1 \cdot f_2, \ldots, f_k)$.

Lemma 2. *The spectrum of a bounded SAP function is the closure of its set of Fourier exponents.*

Lemma 3. *Let f be a complex-valued periodic step function. The set* $\{n \mid n \in Z^+, \hat{f}(n) = 0\}$ *differs from a periodic set in* Z^+ *by at most finitelly many terms.*

The second lemma is a generalization of the analogous result of Bochner and Bohnenblust for Bohr almost periodic functions [3]. A proof of this generalization requiers only minor modifications of the arguments in [3], so we do not reproduce here a proof of Lemma 2, but refer for details to [2].

The third lemma is based on the following result of Lech [7].

(5) If a sequence $(c_\nu)_0^\infty$ of complex numbers satisfies a recursion formula of the type

$$c_\nu = \alpha c_{\nu-1} + \cdots + \alpha_n c_{\nu-n}, \quad \nu \geq n,$$

and if $c_\nu = 0$ for infinitely many values of ν, then those c_ν that are equal to zero occur periodically in the sequance from a certain index on.

We remark that, for the proof of Theorem 3', we need only the following corollary to Lemma 3.

Corollary. Let f be a periodic real-valued non-constant step function. There exist positive integers a and b such that $\hat{f}(an+b) \neq 0$, $n = 0, 1, 2, \ldots$.

To obtain this corollary we need only remark that the set $\{n \mid n \in Z^+, \hat{f}(n) \neq 0\}$ is infinite since f is real-valued, and that an infinite set in Z^+ which differs from a periodic set by at most finitely many terms must contain an infinite arthmetic progression.

4. Proofs

Proof of Theorem 1. Suppose that q has a spectral gap. Then we can find an integrable function b, whose Fourier transform has compact support, for which $s*b\neq 0$ and $q*b=0$. Then, for some real number a,

$$(s*b)\,(a)=\int_{-\infty}^{\infty} s(t)\,b(t-a)\,dt=d\neq 0.$$

Using hypothesis ii), we obtain, since q_n*b is a translate of $q*b=0$,

$$0=\int_{-\infty}^{\infty} q_n(u)\,b(u-a)\,du=\int_{A_n} s(u)\,b(u-a)\,du+\int_{C(A_n)} q_n(u)\,b(u-a)\,du,$$

so that

$$\int_{A_n} s(u)\,b(u-a)\,du=-\int_{C(A_n} q_n(u)\,b(u-a)\,du.$$

Since, as $n\to\infty$, the left side tends to $d\neq 0$, and the right side is, in absolute value, less than

$$\|q\|_{\infty}\int_{C(A_n)} |b(u-a)|\,du,$$

which tends to zero, we have a contradiction.

Proof of Theorem 2. Condition (4) may be written as

$$\int_{0}^{\infty} |\chi_E(t)-\chi_P(t)|\,e^{\alpha t}<\infty,$$

which implies that $F^{-}(\chi_E-\chi_P,\,z)$ is holomorphic on $\operatorname{Im} z<\alpha$. Since the assumption that E is non-periodic means that $\chi_E-\chi_P$ is not almost everywhere equal to a constant, we obtain from (2) that $\chi_E-\chi_P$ has no spectral gap. This ends the proof, since, if χ_E had a spectral gap, it would follow from the discreteness of $\sigma(\chi_P)$ that $\chi_E-\chi_P$ would also have a spectral gap.

Proof of Lemma 1. Let $f=f_1\cdot f_2,\ldots,f_k$. The Fourier series for each f_ν is

$$\sum_{n=-\infty}^{\infty}\widehat{f_\nu}(n)\,e^{i\frac{2n\pi}{p_\nu}x},$$

and the partial sums of these series are

$$S_{\nu,m}(x)=\sum_{n=-m}^{m}\widehat{f_\nu}(n)\,e^{i\frac{2n\pi}{p_\nu}x}.$$

Let $S_m(x)=S_{1,m}(x)\cdot S_{2m}(x),\ldots,S_{k,m}(x)$. We shall show that

$$D_S(S_m,f)\to 0\quad\text{as}\quad m\to\infty. \tag{6}$$

Then, since each S_m is a trigonometric polynomial, f will be a *SAP* function. Since f_ν is of bounded variation, the partial sums $S_{\nu,m}(x)$ are uniformaly bounded, so there exists an M such that $|S_{\nu,m}(x)|\leq M$ and $|f_\nu(x)|\leq M$, $1\leq \nu\leq k$, for all real numbers x, and $m=1, 2, \ldots$. Several applications of the triangle inequality yield

$$D_S(S_m, f)\leq M^{k-1}\sum_{\nu=1}^{k} D_S(S_{\nu,m}, f_\nu).$$

Since $S_{\nu,m}$ tends to f_ν in $L^1(0, p_\nu)$, and $S_{\nu,m}$ and f_ν are periodic with period p_ν, it follows that $D_S(S_{\nu,m}, f_\nu)\to 0$ as $m\to\infty$, so that (6) holds, From (6) we obtain

$$a(f, \lambda)=\lim_{m\to\infty} a(S_m.\ \lambda). \tag{7}$$

An element of $\sigma(f_1)+\sigma(f_2)+\cdots+\sigma(f_k)$ has from

$$\lambda=2\pi\left(\frac{n_1}{p_1}+\frac{n_2}{p_2}+\cdots+\frac{n_k}{p_k}\right),\quad n_\nu\in Z,\ \ \widehat{f_\nu}(n_\nu)\neq 0,\ \ 1\leq \nu\leq k. \tag{8}$$

Since the numbers $p_1^{-1}, p_2^{-2}, \ldots, p_k^{-1}$ are rationally independent, the integers $n_1, n_2, \ldots, n_k$ are uniquely determined by λ. For such a λ, after multiplying the partial sums $S_{\nu,m}$, $1\leq \nu\leq k$, we obtain

$$a(S_m, \lambda)=\widehat{f_1}(n_1)\widehat{f_2}(n_2)\cdots\widehat{f_k}(n_k),$$

for $m\geq\max\{|n_\nu|\,\big|\,1\leq \nu\leq k\}$, and thus, by (7),

$$a(f, \lambda)=\widehat{f_1}(n_1)\widehat{f_2}(n_2)\cdots f_k\,(\widehat{f_k})\,(n_k).$$

Since all the factors on the right are non-zero, $a(f, \lambda)$ is also non-zero, so that, by Lemma 2, $\lambda\in\sigma(f)$. Thus $\sigma(f_1)+\sigma(f_2)+\cdots+\sigma(f_k)$ is contained in $\sigma(f)$, and, since $\sigma(f)$ is closed, our conclusion follows.

Proof of Lemma 3. We can assume that f has a discontinuity at zero. If the period of f is a, then there is a partition $0=x_0<x_1<\cdots<x_{m-1}<x_m=a$ of $[0, a]$ and there are complex numbers c_j, $1\leq j\leq m$, such that

$$f=\sum_{j=1}^{m} c_j\chi_{S_j},\quad S_j=(x_{j-1}, x_j),\quad 1\leq j\leq m,$$

on $(0, a)$. Calculating $\hat f(n)$ for $n\neq 0$, we have

$$\hat f(n)=\frac{i}{2n\pi}\sum_{j=0}^{m-1} d_j e^{-i\frac{2n\pi}{a}x_j}, \tag{9}$$

where $d_j = c_j - c_{j+1}$, $1 \le j \le m-1$, and $d_0 = c_m - c_1$. Letting $D_j = id_j(2\pi)^{-1}$ and $z_j = e^{-i\frac{2\pi}{a}x_j}$, $0 \le j \le m-1$, we have, from (9),

$$n\hat{f}(n) = \sum_{j=0}^{m-1} D_j z_j^n.$$

For each $n \ge 1$, let $g_n = n\hat{f}(n)$. The g_n's are bounded, and we can write, for $|z| < 1$,

$$\sum_{n=1}^{\infty} g_n z^n = \sum_{n=1}^{\infty} \left(\sum_{j=0}^{m-1} D_j (zz_j)^n \right) = \sum_{j=0}^{m-1} D_j \left(\sum_{n=1}^{\infty} (zz_j)^n \right) = \sum_{j=0}^{m-1} \frac{D_j zz_j}{1 - zz_j} = \frac{p(z)}{q(z)},$$

where p and q are polynomials. If $p(z) = a_0 + \cdots + a_k z^k$ and $q(z) = b_0 + \cdots + b_r z^r$ and if $n > \max\{k, r\}$, we have

$$b_0 g_n + b_1 g_{n-1} + \cdots + b_r g_{n-r} = 0.$$

Noting that $b_0 \ne 0$, we then have, again for $n > \max\{k, r\}$

$$g_n = \beta_1 g_{n-1} + \beta_2 g_{n-2} + \cdots + \beta_r g_{n-r},$$

where $\beta_j = -b_j b_0^{-1}$, $1 \le j \le r$. Adplying (5) to the sequence $(g_n)_1^\infty$, we see that the set $\{n \mid n \in Z^+, g_n = 0\}$ is periodic, save for possibly finitely many terms. Since $g_n = n\hat{f}(n)$, the lemma is proved.

Proof of Theorem 3'. Since the f_ν's are of bounded variation, we can apply Lemma 1, and thus it is sufficient to show that the set S of all numbers λ of form (8) is dense on the real line. An application of Lemma 2 will then show that $\sigma(f)$ is the whole real line.

We apply the Corollary to Lemma 2 to the functions f_1 and f_2 so that, for some positive integers a_1, d_1, a_2, and d_2, we have $\widehat{f_1}(a_1 + d_1 r) \ne 0$, $r = 0, 1, 2, \ldots$, and $\widehat{f_2}(-a_2 - d_2 s) \ne 0$, $s = 0, 1, 2, \ldots$. Choose integers $n_3, \ldots, n_k$ such that $\widehat{f_j}(n_j) \ne 0$, $3 \le j \le k$. It follows then that the set S_1 of numbers of the form

$$\lambda = 2\pi \left(\frac{a_1 + d_1 r}{p_1} + \frac{-a_2 - d_2 s}{p_2} + \frac{n_3}{p_3} + \cdots + \frac{n_k}{p_k} \right),$$

where $r = 0, 1, 2, \ldots$, and $s = 0, 1, 2, \ldots$, is a subset of S. Setting $\alpha = 2\pi d_1 p_1^{-1}$, $\beta = 2\pi d_2 p_2^{-1}$, and

$$\gamma = 2\pi \left(\frac{a_1}{p_1} - \frac{a_2}{p_2} + \frac{n_3}{p_3} + \cdots + \frac{n_k}{p_k} \right),$$

we see that S_1 consists of numbers of the from $\lambda = r\alpha - s\beta + \gamma$, where r and s are any non-negative integers. Since p_1^{-1} and p_2^{-2} are rationally independent, so also are α and β. Then S_1, and a fortiori S, is dense on the real line, and the proof is complete.

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