PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 26 (40), 1979, pp. 31–33

ON GENERAL AND REPRODUCTIVE SOLUTIONS OF ARBITRARY EQUATIONS

Dragić Banković

(Received February 19, 1979)

Let S be an arbitrary set and $R \subset S$. R will be called the set of the solutions. Next, let $f: S \to S$ and $F = \{f^{-1}(\{s\}) \mid s \in f(S)\}$, that is, F is quotient class corresponding to the relation $\sim$ of the set S, defined in the following way

$$x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2).$$

Denote by H the set of all mappings $h: S \to S$ with the property

$$(\forall s \in f(S))\ (\exists x \in S)\ (h(x) \in f^{-1}(\{s\})),$$

that is, every class $f^{-1}(\{s\}) \in F$ there is at least one element of the form $h(x)$. Then, it holds.

Proposition 1. *Let $f: S \overset{onto}{\to} R$, i. e. f is the general solution. Then all mappings $g: S \overset{onto}{\to} R$, i. e. all the general solutions can be expressed as follows:*

$$g = fh, \quad \textit{where} \quad h \in H,$$

that is, the equivalence below is true

$$g: S \overset{onto}{\to} R \Leftrightarrow (\exists h \in H)\ (g = fh).$$

Proof: Let g be the general solution. In accordance with the axiom of choice, from any class $f^{-1}(\{y\})$, $y \in R$, we can choose one element. Denote this element by x_y. Define the function h:

$$(\forall s \in R)\ \ (\forall t \in g^{-1}(\{s\}))\ (h(t) = x_s).$$

Then

$$(\forall t \in S)\ (g(t) = s = f(x_s) = f(h(t)).$$

Let us prove that h belongs to the set H. Since g is the general solution, for every $s \in R$ there exists $t \in S$ such that $g(t) = s$ (i. e. $g^{-1}(\{s\})$ is not empty).

Bearing in mind the choice of the element x_s and the definition of the function h, we have for this t

$$h(t)=x_s \text{ and } x_s \in f^{-1}(\{s\}).$$

Conversely, let $g=fh$ for some $h \in H$. Then, for every $t \in S$

$$g(t)=f(h(t))=f(x) \in R, \text{ because } h(t)=x \in S.$$

This means that g is the parametric solution. In accorance with the assumption that $h \in H$, we have

$$(1) \qquad (\forall s \in R)(\exists t \in S)(h(t) \in f^{-1}(\{s\})).$$

Denote by t_s the corresponding, in the sence (1), element for s. Since $h(t) \in \in f^{-1}(\{s\})$, then $f(h(t_s))=s$.

Corollary: *Let f be the general solution. If h is permutation of S then $g=fh$ is the general solution too.*

Proof: Since $(\forall x \in S))\exists t \in S)(x=h(t))$ holds, then $h \in H$.

If the function h which belongs to the set H is chosen "well", one gets the reproductive solution (the solution g is reproductive if $(\forall x \in R)(g(x)=x))$. Namely, the condition for the function h in the above proposition was that for every class $f^{-1}(\{x\})$ there is at least one element $t \in S$ such that $h(t) \in \in f^{-1}(\{x\}$, but in the general case does not have to be $h(x) \in f^{-1}(\{x\})$. If this condition is satisfied, the obtained solution $g=fh$ is reproductive.

Proposition 2. *Let f be the general solution. The mapping $g: S \to S$ is the general reproductive solution if and only if $g=fh$ for some h with the property that for every $x \in R$ $h(x) \in f^{-1}(\{x\})$.*

Proof: Let $g=fh$. Then for every $x \in R$ it is true that

$$g(x)=f(h(x))=x$$

that is, g is reproductive.

Convesely, let g be the general reproductive solution. Determine the function $h|R$ such that for every $x \in R$

$$(h|R)(x) \in f^{-1}(\{x\})$$

which is, in accordance with the axiom of choise, possible. Then for erery $x \in R$

$$f(h(x))=x=g(x).$$

If $x \in S \setminus R$, then

$$(2) \qquad g(x)=s$$

for some $s \in R$, because g is the parametric solution. Since f is the general solution, for every $s \in R$ there existst such that $f(t)=s$, i.e. $t \in f^{-1}(\{s\})$. From every class we choose one such element t, denoting it by t_s. We now define the mapping $h|(S \setminus R)$ such that for every x that satisfies the condition (2)

$$(h|(S \setminus R))(x)=t_s.$$

Then for every $x \in S \setminus R$

$$f(h(x)) = f(t_s) = s = g(x).$$

The proposition 2. is in fact a consequence of the Theorem 2. and Lemma from Reference [1].

REFERENCES

[1] M. Božić, *A note on reproductive solutions*, Publ. Inst. Math. (Beograd) 19 (33) 1976. p. 33—35.

[2] S. Prešić, *Ein Satz uber reproduktive Lösungen*, Publ. Inst. Math. (Beograd) 14 (28), 1972., p. 133—136.

[3] S. Rudeanu, *Boolean Functions and Equations*, North-Holland Amsterdam/London and Elsevier, New York, 1974.

[4] S. Rudeanu, *On reproductive solutions of arbitrary equations* Publ. Inst. Math. (Beograd) 24 (38) 1978. p. 143—145.