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NOTE ON H-CONVEX FUNCTIONS

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Definition 0: *Let H be a nonempty subset of the unit sphere S^{n-1} of n-dimensional Euclidean space R^n. Halfspace $P=\{x \mid \langle a, x\rangle \leqslant \lambda\} \subset R^n$ is called H-convex iff $a \in H$. A set K is said to be H-convex iff it is an intersection of some family of H-convex halfspaces.*

V. G. Boltianskii initiated the examination of H-convex sets and the definition above appeared for the first time in paper [1]. As Boltianskii emphasizes in that paper, the idea for introducing H-convex sets appeared under the influence of papers of P. S. Soltan and his students who have examined class of so called d-convex sets. In the papers [1], [2], [3] of the same author ([3] was written in cooperation with P. S. Soltan) it was shown that classes of H-convex sets and sets related to them had many interesting properties. Among them the central position takes H-convex analogue of Helly's theorem on intersection of convex sets. Under the influence of those papers the author of this note in paper [8] considered a class of H-convex functions and proved, as main, a theorem which is generalization of the theorem of Bohneblust H. F., Karlin S., Shapley L. S. on H-convex functions. The method of the proof of this generalization is elementary and based on standard embedding of R^n into R^{n+1}, $R^n \rightarrow T=\{y \in R^{n+1} \mid \langle e, y\rangle=1\}$ where $e=(0, \ldots, 0, 1) \in R^{n+1}$, and elementary technique of dual cones.

The aim of this paper is to prove, applying the powerful technique given in R. Rockafellar's book [7], strongly strengthened variant of mentioned theorem which is, in fact, H-convex analogue of theorem T. 21.3. of the mentioned book [7]. All of, here, needed definitions and facts about convex functions can be found in book [7]. Also, desired facts about positive and minimal dependence of vectors in R^n can be found in one of the papers [1], [2], [3].

Definition 1. *Affine functional $(\lambda a, \alpha): R^n \rightarrow R$, $x \mapsto \lambda \langle a, x\rangle-\alpha$; $0 \leqslant \lambda$, $\alpha \in R$ is called H-convex if $a \in H$. Closed, proper, convex function f is called H-convex if it is a supremum of some family of H-convex affine functionals.*

Let us introduce the following definition also.

Definition 2. *Closed, proper convex function f is called H^*-convex if f^* is H-convex function (for the definition of the function f^* see [7]).*

Lemma 1. *Let* $\mathcal{A}=\{A_i \mid i\in I\}$ *be a family of H-convex affine functionals provided that* $A_i=(\lambda_i a_i, \alpha_i):R^n\to R$, $x\mapsto \lambda\langle a_i, x\rangle-\alpha_i$ *where* $0\leqslant\lambda_i$ *and* $a_i\in H$ *for every* $i\in I$. *Let* $\mathcal{A}^*=\{A_i^* \mid i\in I\}$ *be a family of corresponding conjugated functionals and* $h=\operatorname{conv}\mathcal{A}^*$. *If* $h(0)<0$ *holds then we can choose functionals* A_1, $A_2, \ldots A_m$ *of family* $\mathcal{A}$, *vectors* $x_1^*, \ldots, x_m^*\in R^n$ *and nonnegative real numbers* $\mu_1, \ldots, \mu_m$ *so that the following conditions are satisfied:*

$$\textstyle\sum_{i=1}^m \mu_i x_i^*=0 \qquad \sum_{i=1}^m \mu_i A_i^*(x_i^*)<0, \qquad m\leqslant md\,H+1$$

(Let us remined ourselves that $md\,H+1$ *is the maximal number of minimally dependent vectors in H.)*

Proof: $h(0)<0$ implies that $(0, -\varepsilon)\in \operatorname{epi}(h)$ for some sufficiently small positive number ε. According to the definition of the function h it can be shown that there exist vectors $x_1^*, \ldots, x_\nu^*$, positive real numbers $\mu_1, \ldots, \mu_\nu$, and convex functionals $A_1^*, \ldots, A_\nu^*$ of family $\mathcal{A}^*$ such that

$$\text{(1)} \qquad \textstyle\sum_{i=1}^\nu \mu_i x_i^*=0 \qquad \sum_{i=1}^\nu \mu_i A_i^*(x_i^*)<0$$

Noticing that $A_i^*=\delta(\cdot\mid\lambda_i a_i)+\alpha_i$ we come up with conclusion that $x_i^*=\lambda_i a_i$ for all $i=1, \ldots, \nu$. So system (1) gets the following form

$$\text{(2)} \qquad \textstyle\sum_{i=1}^\nu \mu_i \lambda_i a_i=0 \qquad \sum_{i=1}^\nu \mu_i \alpha_i<0$$

In case that $\nu\leqslant md\,H+1$ we obtained needed choice. On the contrary, i.e. if $\nu>md\,H+1$ we shall lessen number ν (i.e. shorten the lenghts of sums (2)) relying on some known properties of positive dependent vectors which can be found in anyone of papers [1], [2] [3]. Before that, let us notice that we can assume that $0<\lambda_i$ for all $i=1, \ldots, \nu$ because of the fact that for $\lambda_j=0$ corresponding summand can be omitted from right sum in (2), i.e. summand $\mu_j\alpha_j$, for in case that $\alpha_j<0$, A_j^*, $x_j^*=\lambda_j a_j$ and $0<\mu_j$ already satisfy the condition $\mu_j x_j^*=0$, $\mu_j A_j^*(x_j^*)=\mu_j\alpha_j<0$.

Because of the fact that $md\,H+1<\nu$, vectors a_i, $i=1, \ldots, \nu$, are positively dependent but not minimally dependent. It means that there exists (with appropriate exchange of indices) a sequence $\beta_1, \ldots, \beta_{\nu-1}$ of nonnegative numbers which are not all equall to 0 such that $\beta_1\lambda_1 a_1+\cdots+\beta_{\nu-1}\lambda_{\nu-1}a_{\nu-1}=0$. Now, if $\sum_{i=1}^{\nu-1}\beta_i\alpha_i<0$, the number ν is lessened. Let us assume the contrary, i.e. $\sum_{i=1}^{\nu-1}\beta_i\alpha_i\geqslant 0$. Now, let us observe the sum $\lambda_1(\mu_1-\omega\beta_1)a_1+\cdots+\lambda_{\nu-1}(\mu_{\nu-1}-\omega\beta_{\nu-1})a_{\nu-1}+\lambda_\nu\mu_\nu a_\nu=0$ and let $\omega=\min\{\mu_i\beta_i^{-1}\mid\beta_i>0\}$. So we have shortened the sums in (2) again, preserving their properties. The procedure can be obviously continued until we obtain $\nu\leqslant md\,H+1$. This proves Lemma 1.

The following theorem is the main result of this paper and its proof is a modification of the proof of the theorem T. 21.3. of [7].

Theorem 1. *Let* $\mathcal{F}=\{f_i \mid i\in I\}$ *be a family of H-convex functions in* R^n. *Let C be a nonempty H-convex set in* R^n. *Let functions* f_i *have no recessive direction in common which would be also reccesive direction for the set C. Then, one and only one of two following possibilities is satisfied:*

(a) *there exists a vector* $x\in C$ *such that*

$$f_i(x)\leqslant 0 \text{ for every } i\in I$$

(b) *we can select* $1\leqslant m\leqslant md\,H+1$ *functions from* $\mathcal{F}$, *namely, functions* $f_1,\dots,f_m$, *and find strictly positive real numbers* $\mu_1,\dots,\mu_m$ *so that for some* $\varepsilon>0$ *the following inequality* $\varepsilon\leqslant\sum_{i=1}^{m}\mu_i f_i(x)$ *holds for every* $x\in C$.

Proof. It can be easily verified that $\delta(\cdot\mid C)$ is H-convex function which has the same recessive directions as the set C, which means that adjuncting this function to the family $\mathcal{F}$ we can suppose that $C=R^n$. Let $f_i=\sup\{A_{ij}\mid j\in x_i\}$ where $A_{ij}=(\lambda_{ij}a_{ij},\alpha_{ij})$, $a_{ij}\in H$, $0\leqslant\lambda_{ij}$ for every $i\in I$ and corresponding $j_i\in x_i$. We notice that family $\mathcal{F}$ can be replaced by family $\mathcal{A}=\{A_{ij}\mid i\in I,\ j\in x_i\}$ according to the fact that some direction is recessive for f_i if and only if it is recessive for all functions A_{ij}, $j\in x_i$. Let (a) not be true. We shall show that (b) is satisfied. Let k be positively homogeneous function generated by function $h=\operatorname{conv}\{A_{ij}^*\mid i\in I,\ j\in x_i\}$. The function conjugated to k is the indicator function of convex set $\{x\mid h^*(x)\leqslant 0\}$ (theorem 13.5. of [7]).

Since, according to the theorem 16.5. of [7].

$$h^*=\sup\{A_{ij}^{**}\mid i\in I,\ j\in x_i\}=\sup\{A_{ij}\mid i\in I,\ j\in x_i\}$$

we have that k^* is indicator function of the set $D=\{x\mid A_{ij}(x)\leqslant 0$ for every $i\in I$ and corresponding $j\in x_i\}=\varnothing$, for (a) is not satisfied. Hence $k^*=+\infty$ and $cl(k)=k^{**}=-\infty$ so, especially, $cl(k)(0)=-\infty$. On the same way as in [7], according to the assumption about recessive directions of functions A_{ij} we obtain that $k(0)=clk(0)=-\infty$. Hence, $h(0)<0$. Now, let us apply Lemma 1. So, there exist functionals $A_1,\dots,A_m$ of family $\mathcal{A}$ (for simplicity we have changed indices), vectors $x_1^*,\dots,x_m^*$ and nonnegative real numbers $\mu_1,\dots,\mu_m$ which satisfy conditions:

$$\sum_{i=1}^{m}\mu_i x_i^*=0 \qquad \sum_{i=1}^{m}\mu_i A_i^*(x_i^*)<0, \qquad m\leqslant md\,H+1. \tag{3}$$

If we denote $y_i^*=\mu_i x_i^*$ then (3) can be rewritten in the form

$$\sum_{i=1}^{m}y_i^*=0,\quad \sum_{i=1}^{m}(A_i^*\mu_i)(y_i^*)<0 \quad \text{i.e.} \quad (A_1^*\mu_1\,\square\cdots\square\,A_m^*\mu_m)(0)<0.$$

Hence, according to the theorem 16.4. and 16.1. of [7] we have $A^*=$ $=cl((\mu_1 A_1)^* \square \cdots \square (\mu_m A_m)^*)=cl(A_1^* \mu_1 \square \cdots \square A_1^* \mu_m)$ where $A=\sum_{i=1}^m \mu_i A_i$. So, $A^*(0)<0$ i.e. $\sup\{\langle x, 0\rangle - A(x) \mid x\in R^n\} = -\inf\{A(x) \mid x\in R^n\}<0$ what is equivalent with $0<\inf\{A(x) \mid x\in R^n\}$, which means that there exists $\varepsilon>0$ so that $\varepsilon\leqslant\sum_{i=1}^m \mu_i A_i(x)$ for every $x\in R^n$. So, the proof is finished.

Let us write down H-convex analogue of Consequence 21.3.1. of [7].

Theorem 2. *Let $\mathcal{F}=\{f_i \mid i\in I\}$ be a family of H-convex functions on R^n and C be a nonempty H-convex set in R^n. Let us assume that functions f_i have no recessive direction in common which would be also recessive for C. Let us assume that for every $\varepsilon>0$ and for every set of indices $i_1, \ldots, i_m\in I$, $m\leqslant md\,H+1$, the system of inequalities $f_{i_1}(x)<\varepsilon, \ldots, f_{i_m}(x)<\varepsilon$ has a solution $x\in C$. Then there exists $x\in C$ such that*

$$f_i(x)\leqslant 0 \qquad \text{for every} \qquad i\in I.$$

Proof. The proof can be proceeded applying the Theorem 1 and noticing that the case (b) of the same theorem is impossible.

As a consequence of Theorem 2 we get a form of Helly's theorem for H-convex sets (compare with [2] or [3]).

Theorem 3. *Let $\mathcal{S}=\{C_i \mid i\in I\}$ be a family of nonempty H-convex sets in R^n. Let the sets C_i have no recessive direction in common. If every subset of $\mathcal{S}$ consisted of $m\leqslant md\,H+1$ elements has a nonempty intersection then the family $\mathcal{S}$ has a nonempty intersection also.*

Proof. It is sufficient to apply Theorem 2 on the family of functions

$$\mathcal{S}'=\{\delta(\cdot \mid C_i) \mid i\in I\}$$

Finally, let us notice that other theorems of § 21 of [7], (i.e. theorems T.21.4. and T.21.5.), have their H-convex analogues but we shall not give their formulations here because they are simple modifications of theorems mentioned above.

Let us notice that if the conditions of the Theorem 1 (either Theorem 2 or Theorem 3) are satisfied then the set H can not be contained in a hemisphere of S^{n-1}. Indeed in the last case there would exist unit vector y with property $\langle y, a\rangle\leqslant 0$ for every $a\in H$. However, this would mean that vector y is recessive for every H-function $A=(\lambda a, \alpha)$ and so vector y would be recessive vector for every H-convex function.

Remark: After this paper has been already finished, the book [4] has appeared. Let us notice that the Theorem 3. proved here, answers the question proposed on the page 131 from that book.

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