PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 25 (39), 1979, pp, 27—30

ON NUCLEI OF n-ARY QUASIGROUPS

B. P. Alimpić

(Communicated May 19, 1978)

In this paper we consider a generalization of the concept of nucleus of a groupoid on the n-ary case. If $(Q, \cdot)$ is a groupoid, the left nucleus N_l of $(Q, \cdot)$ is the set of all a in Q such that $(\forall x, y \in Q)\ a\,(xy)=(ax)\,y$. The middle nucleus N_m is similarly defined in terms of $(xa)\,y=x\,(ay)$, and the right nucleus N_r in terms of $(xy)\,a=x\,(ya)$. Obviously, $(N_l, \cdot)$, $(N_m, \cdot)$ and $(N_r, \cdot)$ are semigroups. If $(Q, \cdot)$ is a quasigroup, they are groups.

Let (Q, ω) be an n-ary groupoid. The concept of nucleus may be generalized in different ways. So, H. H. Buchsteiner [2] and Murathudžaev [3] have introduced nuclei for n-ary quasigroup (Q, ω) as certain subsets of the set Q. We define nuclei of an n-ary groupoid (Q, ω) as some subsets of the set Q^{n-1}. This concept of nucleus is a generalization of left and right nuclei of the binary case.

We shall use the following notations. According to V. D. Belousov [1], the sequence $x_k\, x_{k+1} \cdots x_m$ will be denoted by x_k^m. If $k>m$, x_k^m will be considered empty, and if $k=m$, then x_k^m is the element x_k. By $\overset{k}{x}$ we denote the sequence $x \cdots x$ where x is repeated k times.

Let a_1^{n-1} be an arbitrary element of Q^{n-1}, we shall denote it by $\bar{a}$. The mapping $L_i(\bar{a}): Q \to Q$ $(i=1, \ldots, n)$ defined by

$$xL_i(\bar{a}) \overset{\text{def}}{=} a_1^{i-1}\, x a_i^{n-1}$$

is called i-translation of the groupoid (Q, ω) by the sequence $\bar{a}$. Obviously, if (Q, ω) is an n-ary quasigroup, the mappings $L_i(\bar{a})$ are bijections.

D e f i n i t i o n 1. Let (Q, ω) be an n-ary groupoid. The mapping $\lambda: Q \to Q$ with the property

$$(1) \qquad (\forall x_1^n \in Q^n)\ x_1^n\, \omega\lambda = x_1 \lambda \cdots x_{i-1} \lambda\, x_i\, x_{i+1} \lambda \cdots x_n \lambda\omega$$

is called i-regular mapping of the n-ary groupoid (Q, ω).

The set Λ_i of all i-regular mappings of (Q, ω), with respect to the composition $\cdot$ of mappings, is a monoid, and the set Π_i of all i-regular bijections is a group.

From the definition 1 it follows immediately.

Lemma 1. *If (Q, ω) is a quasigroup, then $\Lambda_i = \Pi_i$. In other words, every i-regular mapping of a quasigroup is a bijection.*

Proof. If $\bar{a}$ is an arbitrary element of Q^{n-1}, from (1) we have

$$(\forall\, x \in Q)\, xL_i(\bar{a})\,\lambda = xL_i(\bar{a}\,\lambda),$$

where

$$\bar{a}\,\lambda = a_1\,\lambda \cdots a_{n-1}\,\lambda.$$

Since $L_i(\bar{a})$ and $L_i(\bar{a}\,\lambda)$ are bijections, from $\lambda = L_i(\bar{a}\,\lambda)\,L_i(\bar{a})^{-1}$ it follows that λ is a bijection, too.

Definition 2. Let (Q, ω) be an n-ary groupoid, and N_i $(i = 1, \ldots, n)$ the set of all $\bar{a}$ of Q^{n-1} such that the i-translation $L_i(\bar{a})$ is an i-regular mapping of (Q, ω). The set N_i is called i-nucleus of (Q, ω).

Now we define binary operations $\dot{\circ}_i$, $i = 1, \ldots, n$, of the set Q^{n-1} by the equality

$$\bar{a} \,\dot{\circ}_i\, \bar{b} \overset{\text{def}}{=} \bar{a}\, L_i(\bar{b})$$

where $\bar{a}\,L_i(\bar{b})$ denotes the sequence $a_1 L_i(\bar{b}) \cdots a_{n-1}(\bar{b})$. That is, for example if $n = 3$, $i = 1$: $ab \,\dot{\circ}_1\, cd = acd\,\omega\, bcd\,\omega$.

Lemma 2. *The set N_i is a semigroup with respect to the operation $\dot{\circ}_i$. If λ is an arbitrary i-regular mapping of Q, and $\bar{a} \in N_i$, the sequence $\bar{a}\,\lambda$ belongs to N_i.*

Proof. Let $\bar{a}$, $\bar{b} \in N_i$ and $\bar{x} = x_1^{i-1}\, x_{i+1}^n$. Then we have

$$\begin{aligned}
x_i\, L_i(\bar{x})\, L_i(\bar{a} \,\dot{\circ}_i\, \bar{b}) &= x_i\, L_i(\bar{x})\, L_i(\bar{a}\, L_i(\bar{b})) && \text{(by definition of } \dot{\circ}_i)\\
&= x_i\, L_i(\bar{x})\, L_i(\bar{a})\, L_i(\bar{b}) && (\text{since } \bar{b} \in N_i)\\
&= x_i\, L_i(\bar{x}\, L_i(\bar{a}))\, L_i(\bar{b}) && (\text{since } \bar{a} \in N_i)\\
&= x_i\, L_i(\bar{x}\, L_i(\bar{a})\, L_i(\bar{b})) && (\text{since } \bar{b} \in N_i)\\
&= x_i\, L_i(\bar{x}\, L_i(\bar{a}\, L_i(\bar{b}))) && (\text{since } \bar{b} \in N_i)\\
&= x_i\, L_i(\bar{x}\, L_i(\bar{a} \,\dot{\circ}_i\, \bar{b})) && \text{(by definition of } \dot{\circ}_i).
\end{aligned}$$

Thus we obtain that $\bar{a} \,\dot{\circ}_i\, \bar{b} \in N_i$.

Now we prove associativity of $\dot{\circ}_i$. Let $\bar{a}$, $\bar{b}$, $\bar{c} \in N_i$. We have

$$\begin{aligned}
(\bar{a} \,\dot{\circ}_i\, \bar{b}) \,\dot{\circ}_i\, \bar{c} &= \bar{a}\, L_i(\bar{b}) \,\dot{\circ}_i\, \bar{c} && \text{(by definition of } \dot{\circ}_i)\\
&= \bar{a}\, L_i(\bar{b})\, L_i(\bar{c}) && \text{(by definition of } \dot{\circ}_i)\\
&= \bar{a}\, L_i(\bar{b}\, L_i(\bar{c})) && (\text{since } \bar{c} \in N_i)\\
&= \bar{a} \,\dot{\circ}_i\, \bar{b}_i\, L_i(\bar{c}) && \text{(by definition of } \dot{\circ}_i)\\
&= \bar{a} \,\dot{\circ}_i\, (\bar{b} \,\dot{\circ}_i\, \bar{c}) && \text{(by definition of } \circ_i).
\end{aligned}$$

If $\lambda \in \Lambda_i$ and $\bar{a} \in N_i$, we have

$$x_i L_i(\bar{x}) L_i(\bar{a}\lambda) = x_i L_i(\bar{x}) L_i(\bar{a})\lambda$$
$$= x_i L_i(\bar{x} L_i(\bar{a}))\lambda = x_i L_i(\bar{x} L_i(\bar{a})\lambda) = x_i L_i(x_i L_i(\bar{a}\lambda)),$$

thus $\bar{a}\lambda \in N_i$. This completes the proof of the lemma.

Let $\varphi: N_i \to \Lambda_i$ be the mapping introduced by

$$\varphi\bar{a} \overset{\text{def}}{=} L_i(\bar{a}). \tag{2}$$

By definition of $\circ_i$, we have $xL_i(\bar{a} \circ_i \bar{b}) = xL_i(\bar{a} L_i(\bar{b})) = xL_i(\bar{a}) L_i(\bar{b})$, for every x of Q, thus φ is a homomorphism.

Theorem 1. *If (Q, ω) is an n-ary quasigroup, the nucleus N_i has the following properties:*

(i) *For every $\bar{a} \in N_i$, the sequence $\bar{a} L_i^{-1}(\bar{a})$ is a right identity of $(N_i, \circ_i)$*

(ii) *$(N_i, \circ_i)$ is a regular semigroup.*

(iii) *If $\varphi: N_i \to \Pi_i$ is the mapping defined by (2) the homomorphic image φN_i of N_i is the groupi.*

Proof. (i) By lemma 1, we have $L_i^{-1}(\bar{a}) \in \Pi_i$ and by lemma 2, $\bar{a} L_i^{-1}(\bar{a}) \in$ $\in N_i$. If

$$\bar{b} \in N_i, \quad \bar{b} \circ_i \bar{a} L_i^{-1}(\bar{a}) = \bar{b} L_i(\bar{a} L_i^{-1}(\bar{a})) = \bar{b} L_i(\bar{a}) L_i^{-1}(\bar{a}) = \bar{b},$$

thus $\bar{a} L_i^{-1}(\bar{a})$ is a right identity of $(N_i, \circ_i)$.

(ii) For every $\bar{a} \in N_i$, we observe that $\bar{a} L_i^{-2}(\bar{a}) \circ_i \bar{a} = \bar{a} L_i^{-1}(\bar{a})$.

Thus, $\bar{a} \circ_i \bar{a} L_i^{-2}(\bar{a}) \circ_i \bar{a} = \bar{a} \circ_i \bar{a} L_i^{-1}(\bar{a}) = \bar{a}$, and the semigroup $(N_i, \circ_i)$ is regular.

(iii) If $\bar{a} \in N_i$ and $\lambda \in \Pi_i$, we have $\bar{a} L_i^{-1}(\bar{a})\lambda \in N_i$, and

$$\varphi(\bar{a} L_i^{-1}(\bar{a})\lambda) = \lambda,$$

hence $\varphi N_i = \Pi_i$.

If the n-ary groupoid (Q, ω) has an identity element e, we obtain the following two statements:

Lemma 3. *The element e^{n-1} of Q^{n-1} is a right identity for each of $(N_i, \circ_i)$, $i = 1, \ldots, n$,*

Proof. Indeed, by definition of $\circ_i$, if we put $\bar{e} = e^{n-1}$, we have $\bar{a} \circ_i \bar{e} =$ $= \bar{a} L_i(\bar{e}) = \bar{a}$, because e is an identity of (Q, ω).

Lemma 4. *The monoid $(\Lambda_i, \cdot)$ is a homomorphic image of $(N_i, \circ_i)$, $i = 1, \ldots, n$.*

Proof. Let $\varphi: N_i \to \Lambda_i$ be the mapping introduced by (2). Since φ is a homomorphism, it remains to prove that it is a surjection. If λ is an arbitrary i-regular mapping of (Q, ω), the sequence $\bar{e}\lambda$ belongs to N_i, and $\varphi\bar{e}\lambda = \lambda$. Indeed, since $\lambda \in \Lambda_i$, we have $xL_i(\bar{e}\lambda) = xL_i(\bar{e})\lambda = x\lambda$. The lemma is proved. As a consequence, we have:

Theorem 2. *If* (Q, ω) *is a quasigroup,*

$$N_i \neq \varnothing \iff (\exists \bar{b} \in Q^{n-1})(\forall x \in Q)\, xL_i(\bar{b}) = x.$$

Using a result from [1], we get the following statement.

Theorem 3. *Let* (Q, ω) *be an n-ary quasigroup, with* $n > 3$. *If, for every* $i = 1, \ldots, n$, $N_i = Q^{n-1}$, *there exists a commutative group* $(Q, +)$ *such that*

(i) $(\forall x \in Q)\,(n-2)\,x = 0$.

(ii) $x_1^n \omega = x_1 + \cdots + x_n$.

Proof. If $Q^{n-1} = N_i$, the quasigroup (Q, ω) satisfies i-th Menger's law

$$x_i L_i(\bar{x}) L_i(\bar{y}) = \bar{x}_i L_i(\bar{x} L_i(\bar{y})).$$

By theorem 3.4. of [1], if, for $n > 3$, an n-ary quasigroup (Q, ω) satisfies i-th Menger's law for every $i = 1, \ldots, n$, there exists a commutative group $(Q, +)$ such that (i) and (ii) hold. The theorem is proved.

REFERENCES

[1] В. Д. Белоусов, *n-арные квазигруппы*, „Штиинца", Кишинев, 1972.

[2] H. H. Buchsteiner, *Zentren und Nuclei von n-Loops*, Beiträge zur Algebra und Geometrie, 1, 1971, 85—105.

[3] С. Муратхуджаев, *Группы регулярных подстановок и ядра n-квазигрупп. Сети и квазигруппы*. „Штиинца", Кишинев, 1976.