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ON THE LIMITS OF THE FAMILIES OF LINDENBAUM ALGEBRAS

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A family of theories in the language of the first-order is treated. Correspondig families of Lindenbaum algebras are defined. The relations of the limits of these families are examined.

Let $\mathcal{T}$ be a theory in the language $\mathcal{L}$ of predicate calculus of the first order and let S be a set of formulas in $\mathcal{L}$. A binary relation $\leqslant$ is defined in S, so that for all $p, q \in S$,

$$p \leqslant q \text{ if and only if } \mathcal{T}, p \vdash q.$$

The relation $\leqslant$ is preorder in S. If a binary relation $\sim$ is defined so that for all $p, q \in S$,

$$p \sim q \text{ if and only if } p \leqslant q \text{ and } q \leqslant p$$

then $\sim$ is an equivalence relation on S. The relation $\leqslant$ induces in $S/\sim$ a partial order $\leqslant$ (using the same designation), such that for all $p^c, q^c \in S/\sim$,

$$p^c \leqslant q^c \text{ if and only if } p \leqslant q,$$

where p^c is an equivalence class of p according to $\sim$.

Further $S/\sim$ is designated by P and it is assumed that $\mathbf{P} = \langle P, \leqslant \rangle$ is a directed partial order, i.e. for all $p, q \in P$, there exists $r \in P$ such that $p, q \leqslant r$. For every $p \in P$, $\mathcal{T}_p$ is a deductive closure of $\mathcal{T} \cup \{p\}$. conservative extension of $\mathcal{T}$.

L e m m a 1. *If* $p \leqslant q$, *then* $\mathcal{T}_p \supseteq \mathcal{T}_p$.

P r o o f. Trivial since $q \in \mathcal{F}_p$.

Let $\mathbf{B}_p$ be a Lindenbaum algebra of theory $\mathcal{T}_p, p \in P$. If φ is a formula in the language $\mathcal{L}$ then

$$\varphi_p = \{\psi : \mathcal{T}_p \vdash \varphi \leftrightarrow \psi\}$$

is an element of Lindenbaum algebra $\mathbf{B}_p$. Since $\mathcal{T}_p, p \in P$ is a theory of the first-order, then the algebra $\mathbf{B}_p, p \in P$ is a Boolean algebra.

For all $p, q \in P$, $p \leqslant q$ a mapping $h_{pq}: \mathbf{B}_p \to \mathbf{B}_q$, such that $h_{pq}(\varphi_p) = \varphi_q$, is defined.

Lemma 2. *For all $p, q \in P$, if $p \leqslant q$ then h_{pq} is an embedding.*

Proof. Let $h_{pq}(\varphi_p) = h_{pq}(\psi_p)$, then $\mathcal{T}_q \vdash \varphi \leftrightarrow \psi$. But since $q \in \mathcal{T}_p$, then $\mathcal{T}_p \vdash \varphi \leftrightarrow \psi$, i.e. $\varphi_p = \psi_p$. If $\varphi_p \leqslant \psi_p$, then $\mathcal{T}_p \vdash \varphi \leftrightarrow \psi$, and if $\mathcal{T}_q \vdash \neg(\varphi \leftrightarrow \psi)$, then since $p \leqslant q$, $\mathcal{T}_p \vdash \neg(\varphi \leftrightarrow \psi)$, i.e. $\mathcal{T}_p$ is an inconsistent theory, and lemma holds. If $\mathcal{T}_p$ is consistent, then $\varphi_q \leqslant \psi_q$, i.e. h_{pq} is an embedding.

Since P is a set of formulas in the language $\mathcal{L}$, for all $p, q \in P$, $p \leqslant q$ there exists $\varphi_q \in B_q$, so that $\mathcal{T}_q \vdash \varphi \leftrightarrow p$. Let F be a filter in the algebra $\mathbf{B}_q$ generated with φ_q. F induces a homomorphism $g_{qp}: \mathbf{B}_q \to \mathbf{B}_p$, for all $p \leqslant q$. So we have following.

Lemma 3. *If $p \leqslant q$, then there exists an epimorphism h_{qp} of the algebra $\mathbf{B}_q$ on to $\mathbf{B}_p$.*

Proof. Let ψ_q^F be an element of $B_{q/F}$, where ψ is an arbitrary formula in the language $\mathcal{L}$, then for any formula σ, $\sigma_q \in \psi_q^F$ if and only if $\sigma_q \Leftrightarrow \psi_q \in F$, i.e. $(\sigma_q \Leftrightarrow \psi_q) \leqslant \varphi_q$. Since $\mathcal{T}_q \vdash \varphi \leftrightarrow p$ then $\mathcal{T}_q \vdash p \to (\sigma \leftrightarrow \psi)$. According to $p \leqslant q$, $\mathcal{T}_p \vdash \sigma \leftrightarrow \psi$ and finally $\psi_q^F = \psi_p$.

According the lemma 2, to the family of theories $\mathcal{T}_p, p \in P$ a family $\mathcal{D} = \{\mathbf{B}_p, h_{pq}: p \leqslant q\}$ of Boolean algebras and embeddings is associated. It is obvious that $\mathcal{D}$ is a directed family in the category of Boolean algebras and embeddings [1], [2]. According the lemma 3, to the family $\mathcal{T}_p, p \in P$ is associated an inverse family $\mathcal{T} = \{\mathbf{B}_p, g_{pq}: p \leqslant q\}$ in the category of Boolean algebras and epimorphisms. The following theorem describes the relation of limits of these two families.

Theorem 1. *If $\mathbf{B}_{\mathcal{T}}$ is a Lindenbaum algebra of the theory then* $\varprojlim \mathcal{T} \cong \mathbf{B}_{\mathcal{T}} \cong \varinjlim \mathcal{D}$.

Proof. Let as show that $\mathbf{B}_{\mathcal{T}} \cong \varprojlim \mathcal{T}$. Let B^∞ be a domain of the algebra $\varprojlim \mathcal{T}$. Recall that $B^\infty = \{f \in \prod_{p \in P} B_p : f(p) = g_{qp} f(q),\ p \leqslant q\}$. We sholud notice that for each $f \in B$, there exists a formula φ in the language $\mathcal{L}$ so that $f = \langle \varphi_p : p \in P \rangle$. So, for an arbitrary $p \in P$, $f(p) \in B_p$ and $f(p) = \varphi_p$ for some formula φ in $\mathcal{L}$. Since $\mathbf{P}$ is a directed partial order, so for any $q \in P$, there exist $r \in P$, such $p, q \leqslant r$. According to the definition of homomorphisms g_{qp} we have that $g_{rp}(\varphi_r) = \varphi_p$ and $g_{rq}(\varphi_r) = \varphi_q$, i.e. $f(q) = \varphi_q$ for every $q \in P$.

Define a mapping $g: \varprojlim \mathcal{T} \to \mathbf{B}_{\mathcal{T}}$ such that $g(\langle \varphi_p : p \in P \rangle) = \hat{\varphi}$, where $\hat{\varphi} \in B_{\mathcal{T}}$, i.e. $\hat{\varphi} = \{\psi : \mathcal{T} \vdash \varphi \leftrightarrow \psi\}$. Obviously, g is onto. g is one to one, because if $g(\langle \varphi_p : p \in P \rangle) = g(\langle \psi_p : p \in P \rangle)$ then $\mathcal{T} \vdash \varphi \leftrightarrow \psi$, so for all $p \in P$, $\mathcal{T}_p \vdash \varphi \leftrightarrow \psi$, i.e. $\varphi_p = \psi_p$. Therefore $\langle \varphi_p : p \in P \rangle = \langle \psi_p : p \in P \rangle$.

g is a homomorphism, because if $\langle\varphi_p : p\in P\rangle \leqslant \langle\psi_p : p\in P\rangle$ then for all $p\in P$, $\varphi_p\leqslant\psi_p$. If $\hat{\varphi}\in B_{\mathcal{T}}$, let as define a mapping $\bar{g}_p : \mathbf{B}_{\mathcal{T}}\to\mathbf{B}_p$, so that $\bar{g}_p(\varphi)=\varphi_p$. Obviously $\bar{g}_p$ is a homomorphism so if $\hat{\varphi}\not\leqslant\hat{\psi}$, then $\varphi_p\not\leqslant\psi_q$, contradiction.

To prove that $\varinjlim\mathcal{D}\cong\mathbf{B}_{\mathcal{T}}$, let B_∞ be a domain of the algebra $\varinjlim\mathcal{D}$. Recall that $B_\infty=B/\!\sim$, where $B=\bigcup_{p\in P} B_p\times\{p\}$, and $\sim$ an equivalency on B defined such that for all $\langle a,p\rangle$, $\langle b,q\rangle\in B$, $\langle a,p\rangle\sim\langle b,q\rangle$ if there exists $r\in P$, $p,q\leqslant r$ and $h_{pr}(a)=h_{qr}(b)$.

Mappings $h_p:\mathbf{B}_p\to\varinjlim\mathcal{D}$, $p\in P$ defined such that $h_p(\langle a,p\rangle)=[a,p]$, where $[a,p]$ is an equivalence class according to $\sim$, are embeddings. Instead of $\langle a,p\rangle$, $[a,p]$ we respectively write a_p, $[a_p]$.

Let as define the mapping $h:\varinjlim\mathcal{D}\to\mathbf{B}_{\mathcal{T}}$, such that $h([\varphi_p])=\hat{\varphi}$. h is onto, because for each formula φ in the language $\mathcal{L}$ there exists $\varphi_p\in B_p$, such that $h([\varphi_p])=\hat{\varphi}$. Let $h([\varphi_p])=h([\psi_q])$, $p,q\in P$, then $\mathcal{T}\vdash\varphi\leftrightarrow\psi$, so $\mathcal{T}_p\vdash\varphi\leftrightarrow\psi$, and $\mathcal{T}_q\vdash\varphi\leftrightarrow\psi$. If $r\in P$ such that $p,q\leqslant r$, then $h_{pr}(\varphi_p)=h_{qk}(\varphi_q)$, i.e. $\varphi_p\sim\varphi_q$, so h is one to one.

Let $[\varphi_p]\leqslant[\psi_q]$, then there exists $k\in P$, $p,q\leqslant k$, so that $\varphi_k\leqslant\psi_k$. Let $\bar{h}_k:\mathbf{B}_k\to\mathbf{B}_{\mathcal{T}}$ so that $\bar{h}_k(\varphi_k)=\hat{\varphi}$. $\bar{h}_k$ is an embedding because, if $\hat{\varphi}=\hat{\psi}$ then $\mathcal{T}\vdash\varphi\leftrightarrow\psi$, i.e. $\mathcal{T}_k\vdash\varphi\leftrightarrow\psi$, so $\varphi_k=\psi_k$ and $\bar{h}_k$ is one to one. If $\varphi_k\leqslant\psi_k$, then $\mathcal{T}_k\vdash\varphi\to\psi$, and so if $\mathcal{T}\vdash\neg(\varphi\to\psi)$, then $\mathcal{T}_k\vdash\neg(\varphi\to\psi)$ and $\mathcal{T}_k$ is an inconsistent theory, so the theorem trivially holds. If $\mathcal{T}_k$ is consistent then $\mathcal{T}\vdash\varphi\to\psi$, i.e $\hat{\varphi}\leqslant\hat{\psi}$ and $\bar{h}_k$ is an embedding. Therefore h is a homomorphism and the theorem is proved.

Further are considered the families $\overline{\mathcal{D}}$, $\overline{\mathcal{T}}$ dual to the families $\mathcal{D}$, $\mathcal{T}$. More precisely, the relation $\geqslant$ dual to the relation $\leqslant$ is considered. In that way the direct family $\mathcal{D}$ is transmitted in an inverse family $\overline{\mathcal{D}}=\{\mathbf{B}_p, h_{pq}: p\geqslant q\}$ of Boolean algebras and embeddings. Similarely, the family $\mathcal{T}$ is transmitted in to a direct family $\overline{\mathcal{T}}=\{\mathbf{B}_p, g_{qp}: p\geqslant q\}$ of Boolean algebras and epimorphisms. The following theorem is investigating the relation of limits of dual families.

Let $\mathcal{T}_P=\bigcup_{p\in P}\mathcal{T}_p$ and let $\mathbf{B}_P$ be a Lindenbaum algebra of the theory $\mathcal{T}_P$.

Theorem 2. $\varprojlim\overline{\mathcal{D}}\cong\mathbf{B}_P\cong\varinjlim\overline{\mathcal{T}}$.

Proof. Let as prove that $\mathbf{B}_P\cong\varinjlim\overline{\mathcal{T}}$. Let $\bar{g}:\varinjlim\mathcal{T}\to\mathbf{B}_P$ defined by $\bar{g}([\varphi_p])=\varphi_P$. It is obvious that for any formula φ in the language $\mathcal{L}$ there exists $p\in P$ so that $\varphi_p\in B_p$, i.e. $\bar{g}$ is onto. Let $\bar{g}([\varphi_p])=\bar{g}([\psi_p])$, $p,q\in P$.

Hence $\mathcal{T}_P \vdash \varphi \leftrightarrow \psi$. Since the proof for $\varphi \leftrightarrow \psi$ in the theory $\mathcal{T}_P$ is final, there exists a final sequence $p_1, \ldots, p_n \in P$ so that $\mathcal{T}, p_1, \ldots, p_n \vdash \varphi \leftrightarrow \psi$. Let $r \in P$ so that $p_1, \ldots, p_n \leqslant r$, such r exists since $\mathbf{P}$ is directed, and let $\bar{g}_r : \mathbf{B}_r \to \mathbf{B}_P$, be a mapping defined so that $\bar{g}_r(\varphi_r) = \varphi_P$. We should notice that $\bar{g}_r$ is an embedding, because if $\varphi_r \neq \psi_r$ and $\varphi_P = \psi_P$ then $\mathcal{T}_P \vdash \varphi \leftrightarrow \psi$, so $\mathcal{T}, p_1, \ldots, p_n \vdash \varphi \leftrightarrow \psi$. But $\mathcal{T}_r \vdash \neg(\varphi \leftrightarrow \psi)$ and $p_1, \ldots, p_n \leqslant r$, so that $\mathcal{T}, p_1, \ldots, p_n \vdash \neg(\varphi \leftrightarrow \psi)$ i.e $\mathcal{T}_P$ is an inconsistent theory and theorem trivialy holds. If $\mathcal{T}_P$ is consistent then $\varphi_P \neq \psi_P$, so $\bar{g}_r$ is one to one, and hence $\bar{g}$ is one to one. It is obvious that $\bar{g}_r$ is a homomorphism, hence $\bar{g}$ is a homomorphism too.

To prove $\mathbf{B}_P \cong \varprojlim \overline{\mathcal{D}}$, we have to notice that the domain $\bar{B}^\infty$ of the algebra $\varprojlim \overline{\mathcal{D}}$, $B^\infty = \{f \in \prod_{p \in P} B_p : f(q) = h_{pq} f(p), p \geqslant q\}$. Every $f \in \bar{B}^\infty$ is of the form $\langle \varphi_p : p \in P \rangle$, where φ is a formula in the language $\mathcal{L}$. Further on, we should procede in essentially the same way as in the proof of the first part of Theorem 1.

REFERENCES

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