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ORLICZ-PETTIS TOPOLOGIES IN FUNCTION SPACES

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The Mikusinski Diagonal Theorem ([4]) has found many applications in abstract functional analysis and measure theory. In this note we present yet another application to Orlicz-Pettis type theorems. Let S be a compact Hausdorff space and $C(S)$ the space of all continuous real-valued functions on S (for convenience, we assume real scalars throughout). G.E.F. Thomas has shown that a series $\sum f_m$ in $C(S)$ which is subseries convergent (s.s. convergent) with respect to the topology of pointwise convergence is s.s. convergent with respect to the sup-norm topology ([6] II. 4). (A series $\sum x_n$ in a locally convex space (E, τ) is s.s. convergent with respect to τ if for each subsequence $\{x_{m_k}\}$, the series $\sum x_{m_k}$ is τ-convergent.) In Theorem 2 we use a form of the Mikusinski Diagonal Theorem due to P. Antosik ([1]) to give a very simple proof of this fact.

In [7], I. Tweddle established a generalization of the classical Orlicz-Pettis Theorem for locally convex spaces. If (E, τ) is a locally convex space, Tweddle showed that there is a strongest locally topology on E, denoted by $OP(\tau)$, such that each τ-s.s. convergent series in E is $OP(\tau)$-s.s. convergent. In corollary 3, we use Theorem 2 to derive a part of Tweddle's result. If p denotes the pointwise topology on $C(S)$, we also observe that $OP(p)$ is the sup-norm topology.

Finally, we observe that the analogue of the result of Thomas discussed above is not valid for the topology of pointwise convergence on the space of bounded measurable functions. If p denotes the pointwise topology on the space of bounded measurable functions, in Theorem 4 we describe $OP(p)$.

For later reference we state Diagonal Theorem of Antosik ([1]). The result is valid for normed groups, but we only require the result for Banach spaces.

Diagonal Theorem. Let X be a Banach space and $x_{ij} \in X$. If $\lim_j x_{ij} = o$ for each i, then there exists an infinite subset I of N *and a subset J of* N *such that (i) $\sum_{j \in J} |x_{ij}| < \infty$ and (ii) $|\sum_{j \in J} x_{ij}| \geq |x_{ii}|/2$ for al $i \in I$.*

Also for later reference we recall the following basic result for subseries convergence.

Lemma 1. *Let τ and σ be two locally convex topologies on the vector space E with σ weaker than τ. If every σ-s.s. convergent series has a subsequence which is τ-convergent to 0, then every σ-s.s. convergent series is τ-s.s. convergent.*

For the proof see the last paragraph of page 60 of [2] or II. 1 of [6].

Theorem 2. *Let $\sum f_m$ be s.s. convergent with respect to the topology of pointwise convergence in $C(S)$. Then $\sum f_m$ is s.s. convergent with respect to the sup-norm.*

Proof. First assume S is metrizable. For each m there is a $t_m \in S$ such that $|f_m(t_m)| = \|f_m\|$. Now there exists a subsequence $\{t_{m_k}\}$ which converges to $t \in S$. The subsequence $\{\|f_{m_k}\|\}$ converges to 0; for if this were not the case, there would exist $\varepsilon > o$ and a subsequence (which for convenience of notation we assume is $\{m_k\}$) such that $|f_{m_k}(t_{m_k})| = \|f_{m_k}\| \geq \varepsilon$. Set $Z_{ij} = f_{m_j}(t_{m_i}) - f_{m_j}(t)$. Then $\lim_j Z_{ij} = o$ for each i by the pointwise s.s. convergence of $\sum f_m$. By the Diagonal Theorem $|\sum_{j \in J} Z_{ij}| \geq |Z_{ii}|/2$ for each $i \in I$ (where we use the notation of the Diagonal Theorem). Hence

$$(1) \qquad \Big|\sum_{j \in J} f_{m_j}(t_{m_i}) - \sum_{j \in J} f_{m_j}(t)\Big| \geq |f_{m_i}(t_{m_i}) - f_{m_i}(t)|/2 \geq$$

$$\{|f_{m_i}(t_{m_i})| - |f_{m_i}(t)|\}/2 \geq \varepsilon/2 - |f_{m_i}(t)|/2 > \varepsilon/4$$

for large i since $f_{m_i}(t) \to o$. But (1) implies $\sum_{j \in J} f_{m_j}$ is not continuous since $t_{m_i} \to t$. Thus $\|f_{m_k}\| \to o$, and $\sum f_m$ is s.s. convergent with respect to the sup-norm by Lemma 1.

If S is not metrizable, we use the method of Thomas. Define an equivalence relation $\sim$ on S by $t \sim s$ iff $f_n(s) = f_n(t)$ for each n. Let S_0 be the equivalence classes of S under $\sim$ with $\pi: S \to S_0$ the quotient map which associates with each $s \in S$ the equivalence class determined by s. Now S_0 is compact, metrizable under the metric $d(\pi s, \pi t) = \sum_n |f_n(s) - f_n(t)|/2^n$, $s, t \in S$ (see the proof of VI. 7.6 in [3]}. Define continuous functions $F_n: S_0 \to \mathbf{R}$ by $F_n(\pi(s)) = f_n(s)$. Now $\sum F_n$ is s.s. convergent with respect to the topology of pointwise convergence on S_0 and by the above is s.s. convergent with respect to the topology of uniform convergence on S_0. Thus $\sum f_n$ is s.s. convergent with respect to the topology of uniform converence on S.

Let (E, τ) be a locally convex Hausdoff space. Let S be the family of all series in E which are $\sigma(E, E')$ — s.s. convergent, and let $E^{\#}$ be the set of all linear functionals x' on E such that $\sum \langle x', x_m \rangle = \langle x', \sum x_m \rangle$ for all series $\sum x_m$ in S. Thus, $E' \subseteq E^{\#}$.

Corollary 3 (Tweddle). *Let $\sum x_m$ be $\sigma(E, E')$-s.s. convergent in E. Then $\sum x_m$ is s.s. convergent with respect to the to the topology of uniform convergence* on *$\sigma(E^{\#}, E)$ compact subset of $E^{\#}$, i.e., $\sum x_m$ is s.s. convergent with respect to the Mackey topology $\tau(E, E^{\#})$.*

Proof. Let $K \subseteq E^{\#}$ be $\sigma(E^{\#}, E)$ compact. Each x_m is a continuous function on K when K has the $\sigma(E^{\#}, E)$ topology so $x_m \in C(K)$ and $\sum x_m$ is s.s. convergent with respect to the pointwise topology in $C(K)$. By Theorem 2, $\sum x_m$ is s.s. convergent with respect to the sup-norm topology in $C(K)$, i.e., $\sum x_m$ is s.s. convergent uniformly on K.

Since $E' \subseteq E^{\#}$, the topology $\tau(E, E^{\#})$ is stronger that the Mackey topology $\tau(E, E')$ so Tweddle's result generalizes the classical Orlicz-Pettis Theorem ([2] IV. 1). If τ is a locally convex topology on the vector space E, the strongest locally convex topology τ' on E such that each τ-s.s. convergent series is τ-s.s. convergent is called the Orlicz-Pettis topology on E with respect to τ and denoted by $OP(\tau)$. Tweddle shows that $OP(\tau)=OP(\sigma(E, E'))=$ $=\tau(E, E^{\#})$, where $E'=(E, \tau)'$ ([7], Prop. 1).

From Theorem 2, if p denotes the topology of pointwise convergence on $C(S)$, it follows that $OP(p)$ is stronger than the sup-norm topology on $C(S)$. But it follows from the remark after Proposition 2 of [7] that $OP(\sigma(C(S), C(S)'))$ is the sup-norm topology. Since $OP(p) \subseteq OP(\sigma(C(S), C(S)'))$, $OP(p)$ must be exactly the sup-norm topology.

It is of interest to note that a result analogous to Theorem 2 does not hold for the space of bounded measurable functions. That is, the continuity of the functions in Theorem 2 is important: the continuity is clearly used in the proof. Let $\sum$ be a σ-algebra of subsets of a set S and let $B(S, \sum)$ be the space of all real-valued, $\sum$ — measurable functions on S. If $\{E_j\}$ is a disjoint sequence from $\sum$, then the series $\sum C_{E_j}$, where C_E is the characteristic function of a set E, is s.s. convergent with respect to the topology of pointwise convergence on S but is clearly not s.s. convergent with respect to the sup-norm. Thus, if p denotes the pointwise topology on $B(S, \sum)$, then $OP(p)$ is strictly weaker than the sup-norm topology in contrast to the situation for $C(S)$. We can, however, give a description of $OP(p)$. We denote by $ca(\sum)$ the space of all real-valued, countably additive functions on $\sum$. Recall $ca(\sum)$ is a closed subspace of the dual of $B(S, \sum)$ under the sup-norm topology ([3] IV. 5. 1).

Theorem 4. *$OP(p)$ is the Mackey topology $\tau(B(S, \sum), ca(\sum))$.*

Proof. By Tweddle's description of $OP(p)$ it is necessary to identify the subspace $E^{\#}$ of the algebraic daul of $B(S, \sum)$ with the property that $\sum \phi_m$ p-s.s. convergent in $B(S, \sum)$ implies $\sum \gamma(\phi_m)=\gamma(\sum \phi_m)$ for $\gamma \in E^{\#}$. Let $\{E_j\}$ be a disjoint sequence from $\sum$. The series $\sum C_{E_j}$ is p-s.s. convergent to C_{UE_j} so if $\gamma \in E^{\#}$, then $\sum \gamma(E_j)=\gamma(UE_j)$, i.e., $\gamma \in ca(\sum)$ or $E^{\#} \subseteq ca(\sum)$. We show $E^{\#}=ca(\sum)$.

Let $\sum \phi_n$ be subseries convergent in $B(S, \sum)$ with respect to the topology of pointwise convergence. If F denotes the family pf all finite subset of the positive integers **N**, then the "partial sums" $\{\sum_{n \in \sigma} g_n : \sigma \in F\}$ are norm-bounded in $B(S, \sum)$. For if this were not the case, for each k there would exist $\sigma_k \in F$, $t_k \in S$ such that

$$|\sum_{n \in \sigma_k} \phi_n(t_k)| > k. \tag{2}$$

Since $\sum \phi_n$ is subseries convergent in $B(S, \Sigma)$ with respect to the topology of pointwise convergence, the sequence $\{\phi_n(t_k)\}_{k=1}^{\infty} \in l^{\infty}$ is subseries convergent in l^{∞} with respect to p. Thus, for each k we may define a countably additive measure μ_k on the family P of all subsets of N by $\mu_k(\sigma) = \sum_{n \in \sigma} \phi_n(t_k)$ with $\sup_k |\mu_k(\sigma)| < \infty$ for each $\sigma \in P$. By the Nikodym Boundedness Theorem ([3] IV.9.8; cf [5] for a proof based on the diagonal theorem), $\sup\{|\mu_k(\sigma)| : k, \sigma \in P\} < \infty$ contradicting (2).

If $\{n_k\}$ is any subsequence, the partial sums $\left\{\sum_{j=1}^{n} \phi_{n_j}\right\}$ are uniformly bounded so if $\nu \in ca(\Sigma)$, the Bounded Convergence Theorem implies $\lim_k \sum_{j=1}^{k} \nu(\phi_{n_j}) = \nu\left(\sum_{j=1}^{\infty} \Phi_{n_j}\right)$. Thus $\gamma \in E^{\#}$, and by Tweddle's result, $OP(p) = \tau(B(S, \Sigma), ca(\Sigma))$.

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