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ON π-GROUPOIDS

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In this paper n-ary π-groupoids are considered. Some properties of partial isotopy of π-groupoids and quasigroups are given and proper π-groupoids are investigated. A theorem which gives necessary and suffficient conditions for the completion of one kind of partial latin hyper-cubes is proved, these conditions are necessary and sufficient for a π-groupoid to be proper. Theorems which establish connections between isotopy and partial isotopy are given.

Binary π-groupoids and π-nets, which appeared in connection with geometrical interpretation of closure conditions which correspond to identities with parameters, were considered in [1].

We shall use notations and terminology from [3].

Definition 1. An n-ary groupoid (G, A) is called an n-ary π-groupoid (n-π-groupoid) if and only if there exists $(a_1^n)\in G^n$ such that for every $i\in\{1, 2, \ldots, n\}$ the following conditions hold:

(i) If $A(a_1^{i-1}, u, a_{i+1}^n)=A(a_1^{i-1}, v, a_{i+1}^n)$ then $u=v$.

(ii) If $(x_1^{i-1}, x_{i+1}^n)\neq(a_1^{i-1}, a_{i+1}^n)$, $u\neq a_i$, $v\neq a_i$,

then from $A(x_1^{i-1}, u, x_{i+1}^n)=A(x_1^{i-1}, v, x_{i+1}^n)$ follows $u=v$.

The ordered n-tuple (a_1^n) we call marked and the element $A(a_1^n)=a_0$ we call a knot element.

In the sequel we shall consider only finite n-π-groupoids.

The elements $A(a_1^{i-1}, x, a_{i+1}^n)$, $x\in G$, belong to one line of the Cayley hyper-cube of n-π-groupoid A with marked n-tuple (a_1^n) and this line we call marked. The n-π-groupoid A has n marked lines, the knot element a_0 appears in the intersection of these lines.

If in the Cayley hyper-cube of the n-π-groupoid A we delete elements in marked lines, we get a partial hyper-cube which we call the remainder of the given Cayley hyper-cube for the marked n-tuple (a_1^n). From Definition 1, (ii), it follows that the partial hyper-cube is latin.

To illustrate these definitions we give examples of finite n-π-groupoids for $n=2$, 3 and state some properties of Cayley table and Cayley cube of these π-groupoids.

	1	2	3	4
1	1	4	2	3
2	4	1	1	2
3	1	2	3	4
4	2	3	4	1

This is a Cayley table of a binary π-groupoid of order 4 in which (3,2) is a marked pair. The third row and the second column of the table are permutations of the set on which the π-groupoid is defined (this follows from Definition 1, (i)). If in this table we delete this row and column we get 3×3 table, the remainder of the given Cayley table for the marked pair (3,2). From Definition 1, (ii), it folows that no element can appear twice in any row or column of the remainder, hence, the remainder is a partial latin square. If an element appears twice in a row (column) of the complete Cayley table, then it appears once in the intersection of this row (column) and the marked column (row).

Of cource, a π-groupoid can have more than one marked pair. In this example (3,3), (1,2) and (1,3) are also marked pairs.

We giwe also an example of a ternary π-groupoid:

A_1	1	2	3	4
1	3	①	1	2
2	4	1	2	3
3	1	2	3	4
4	2	3	4	1

A_2	1	2	3	4
1	4	②	2	3
2	1	2	3	4
3	2	3	4	1
4	3	4	1	2

A_3	1	2	3	4
1	2	③	1	4
2	2	1	4	1
3	3	2	1	2
4	4	4	2	3

A_4	1	2	3	4
1	2	④	4	1
2	3	4	1	2
3	4	1	2	3
4	1	2	3	4

This 3-π-groupoid A is represented by four binary operations $A_i(x, y) = A(i, x, y)$, $i=1, 2, 3, 4$. The marked triple is (3, 1, 2). Elements in circles form marked 1-line*) and framed elements in table for A_3 form marked 2-line and marked 3-line.

An n-ary quasigroup (Q, A) is an n-ary π-groupoid in which every ordered n-tuple of elements from Q is marked.

A natural question is what is the relation between quasigroups and π-groupoids? One way to obtain quasigroups from some π-groupoids is to make suitable permutations of elements in marked lines. For example, if we consider the 3-π-groupoid from the preceeding example and instead of the circled elements 1, 2, 3, 4 we put the elements 4, 1, 2, 3 respectively, and in A_3 table instead of the first row (2, 3, 1, 4) we put (1, 2, 3, 4), instead of the

*) In the Cayley hyper-cube of an n-π-groupoid A the line which consists of elements $A(b_1^{i-1}, x, b_{i+1}^n)$, $x\in G$, we call i-line. For $n=2$ columns of the Cayley table are 1-lines and rows are 2-lines.

second column (3, 1, 2, 4) we put (2, 3, 4, 1), we shall get a ternary quasigroup.

This suggests the following definition.

Definition 2. Let (G, A) be an n-π-groupoid with the marked ordered n-tuple (a_1^n). An n-groupoid (G, B) is partially isotopic with respect to (a_1^n), to the n-π-groupoid (G, A) if and only if for every $i \in \{1, 2, \dots, n\}$

$$B(x_1^n) = \begin{cases} \alpha_i A(a_1^{i-1}, x_i, a_{i+1}^n), \text{ for } (x_1^{i-1}, x_{i+1}^n) = (a_1^{i-1}, a_{i+1}^n), \\ A(x_1^n), \text{ for other values of } (x_1^n), \end{cases}$$

where α_1, $i = 1, 2, \dots, n$ are prermutations of the set G such that

$$\alpha_1 a_0 = \alpha_2 a_0 = \cdots = \alpha_n a_0,$$

and a_0 is the knot element $(A(a_1^n) = a_0)$ of the n-π-groupoid (G, A).

That (G, B) is partially isotopic to (G, A) with respect to $(a_1^n) \in G^n$ we shall denote by $B = A^P$, $P = (\alpha_1^n, a_1^n)$.

From Definition 2 we get that the partial isotopy has the following properties:

1° Partial isotope of an n-π-groupoid is also an n-π-groupoid. These two n-π-groupoids have the same marked n-tuples.

2° A partial isotopy with respect to one fixed marked n-π-tuple (a_1^n) is an equivalence relation in the set of all n-π-groupoids with the marked n-tuple (a_1^n) defined on the same nonempty set G.

If we cosider partial isotopies with respect to different marked n-tuples then transitivity need not hold.

3° If two n-quasigroups are partialy isotopic then they coincide.

Definition 3. An n-π-groupoid with the marked n-tuple (a_1^n) which is partially isotopic with respect to (a_1^n) to an n-quasigroup is called proper with respect to (a_1^n).

A criterion which gives necessary and sufficient conditions for an n-π-groupoid to be proper is given by the following theorem. This theorem also gives necessary and sufficient conditions for a completion of one kind of partial latin hyper-cubes.

Theorem 1. *Let (G, A) be an n-ary π-groupoid of order m with marked n-tuple (a_1^n) and let $N(x)$ denotes the number of appearances of element $x \in G$ in the remainder for (a_1^n) of the Cayley hyper-cube of this n-π-groupoid. The n-π-groupoid (G, A) is proper with respect to (a_1^n) if and only if there exists an element $x_0 \in G$ such that $N(x_0) = m^{n-1} - 1$ and for every $y \in G$, $y \neq x_0$, $N(y) = m^{n-1} - n$.*

Proof. Suppose that there exists an element $x_0 \in G$ such that $N(x_0) = m^{n-1}-1$ and for every $y \in G$, $y \neq x_0$ $N(y) = m^{n-1}-n$. We shall prove that then (G, A) is partially isotopic to an n-quasigroup.

If in the Cayley hyper-cube of the given n-π-groupoid A we delete elements in the marked lines, we get a partial latin hyper-cube. We shall prove that this partial latin hyper-cube can be completed to n-dimensional latin hyper-cube of order m (i. e. Cauley hyper-cube of an n-quasigroup of order m).

This completion we shall perform in the following way. First, in the intersection of the marked lines (whose cells are now empty) we put the element x_0.

Now we consider $n(m-1)$ binary groupoids which are obtained from A fixing all but $n-2$ variables in the following way:

$$(1)\qquad \begin{cases} A^{(1)}_{p_1}(x, y) = A(p_1, x, y, a_4, \dots, a_n),\ p_1 \in G,\ p_1 \neq a_1, \\ A^{(2)}_{p_2}(x, y) = A(x, p_2, y, a_4, \dots, a_n),\ p_2 \in G,\ p_2 \neq a_2, \\ A^{(3)}_{p_3}(x, y) = A(x, y, p_3, a_4, \dots, a_n),\ p_3 \in G,\ p_3 \neq a_3, \\ A^{(i)}_{p_i}(x, y) = A(x, y, a_3, \dots, a_{i-1}, p_i, a_{i+1}, \dots, a_n), \\ p_i \in G,\ p_i \neq a_i,\ i = 4, \dots, n. \end{cases}$$

After deleting all elements in the marked lines in Cayley hyper-cube of A, the Cayley table of each of the binary groupoids (1) becomes a partial latin square with exactly one cell empty. It is not difficult to see that every such partial latin square can be uniquely completed to latin square of order m (i. e. Cayley table of a binary quasigroup). So, in the empty cell of each of these partial latin squares we put the element which completes partial latin square to latin square.

Now we have filled all empty cells.

We shall prove that in this way every other partial latin square which had exactly one cell empty (obtained from A fixing $n-2$ variables) is completed to a latin square. Let

$$(2)\qquad \bar{A}(x, y) = A(a_1, \dots, a_k, x, a_{k+1}, \dots, a_{j-1}, y, a_{j+1}, \dots, a_{i-1}, p_i, a_{i+1}, \dots, a_n)$$

be such a binary groupoid whose Cayley table had exactly one cell empty. Groupoids

$$A^{(i)}_{p_i}(x, y) = A(x, y, a_3, \dots, a_{i-1}, p_i, a_{i+1}, \dots, a_n)$$

and

$$\bar{\bar{A}}(x, y) = A(x, a_2, \dots, a_{j-1}, y, a_{j+1}, \dots, a_{i-1}, p_i, a_{i+1}, \dots, a_n)$$

have one common line which contained empty cell, which means that the element which we have put in that cell to complete partial groupoid obtained from $A^{(i)}_{p_i}$ to a quasigroup, also completes the partial proupoid obtained from $\bar{\bar{A}}$ to a quasigroup. The same is true for groupoids $\bar{\bar{A}}$ and $\bar{A}$, and since the empty cell was common for all three groupoids we get that when we completed the partial groupoid obtained from $A_{p_i}^{(i)}$ to a quasigroup, then the partial groupoid obtained from $\bar{A}$ was also completed to a quasigroup.

We are going to prove that this completion gives a hyper-cube of order m which is latin, i. e. a Cayley hyper-cube of an n-quasigroup. The n-ary groupoid defined by this Cayley hyper-cube we denote by B and its lines obtained from the marked lines of the n-π-groupoid A we shal also call marked.

Partial latin hyper-cube has $m^{n-1}-1$ 1-lines. Since $N(x_0)=m^{n-1}-1$ we get that x_0 appears exactly once in every 1-line of partial latin hyper-cube. The same is true for i-lines, $i=2, 3, \ldots, n$.

This means that x_0 appears in every row and every column of partial latin squares obtained from groupoids (1), hence every element by which these partial latin squares were completed was different from x_0. x_0 was put in the intersection of marked lines, thus in the Cauley hyper-cube of B x_0 appears exactly m^{n-1} times, in every i-line, $i=1, 2, \ldots, n$ exactly once.

In the partial latin hyper-cube every line with exactly one empty cell belongs to a Cayley table of one binary groupoid (2). Thus, in the Cayley hyper-cube of B in every line which is not marked no element appears twice.

Now we shall prove that also in every marked line there is no element which appears twice.

Let us suppose the contary, i.e. an element $y_0 \in G$ appears at least twice in one marked line, let's say in the marked 1-line. y_0 apears m^{n-1}-n times in the partial latin hyper-cube and at least twice in the marked 1-line, thus in the partial latin hyper-cube to which the marked 1-line is added y_0 appears at least $m^{n-1}-n+2$ times.

The partial latin hyper-cube has $m^{n-1}-1$ 2-lines, and among these lines at least $m^{n-1}-n+2$ contain one y_0, so in the partial latin hyper-cube there exist at most n-3 2-lines without y_0. This means that y_0 can appear in at most n-3 marked i-lines, $i=3, 4, \ldots, n$. But there are n-2 these marked lines, so at least one of these marked lines is without y_0. Suppose that the marked 3-line is without y_0.

The Cayley hyper-cube of B has m^{n-1} 3-lines, in the marked 3-line there is no y_0, in every other 3-line y_0 appears once, thus in the whole Cayley hyper-cube of B y_0 appears $m^{n-1}-1$ times.

Since y_0 in the Cayley hyper-cube of B appears $m^{n-1}-1$ times there exists an element $z_0 \in G$ which in this hyper-cube appears at least $m^{n-1}+1$ times (then $z_0 \neq x_0$). We have n marked lines and $N(z_0)=m^{n-1}-n$, this means that in the n marked lines z_0 appears at least $n+1$ times, so z_0 appears twice in one marked line.

From here, in the same way as we have done it for y_0, we get that z_0 appears in the Cayley-hyper-cube of B $m^{n-1}-1$ times, which is the contradiction.

So, we have proved that no element can appear twice in any line of the Cayley hyper-cube of the finite n-groupoid B which means that B is an n-quasigroup. From the way we obtained B from A it follows that the n-quasigroup B is partially isotopic to the n-π-groupoid A.

This proves one half of the theorem, the other half is obvious, so the theorem is proved.

Remark 1. The proof of the preceeding theorem is given in the most general form because the proof of the theorem for $n=3$, 4 does not require all the arguments which are necessary in the general case.

Remark 2. This theorem generalizes to n-dimensional case one special case of Ryser's theorem on embedding latin rectangles in latin squares [2].

Now we give an example of a ternary π-groupoid A of order 4 with the marked triple (4, 1, 1) which is not proper (which can be established using Theorem 1).

A_1	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

A_2	1	2	3	4
1	2	3	4	2
2	3	2	1	4
3	4	1	2	3
4	2	4	3	1

A_3	1	2	3	4
1	3	1	2	3
2	1	4	3	2
3	2	3	4	1
4	3	2	1	4

A_4	1	2	3	4
1	4	2	1	3
2	1	1	2	3
3	3	2	3	4
4	2	3	4	2

We give now two theorems about ordinary isotopy and partial isotopy. Proofs of these theorems are omited since they are analogous to the proofs of the corresponding theorems in binary case.

Theorem 2. *Let (G, A) be an n-π-groupoid with marked n-tuple (a_1^n). If an n-groupoid (G, B) is isotopic to the n-π-groupoid (G, A) $B=A^T$, $T=(\alpha_1^n, \beta)$, then B is an n-π-groupoid with marked n-tuple $(\{\alpha_j^{-1} a_j\}_{j=1}^n)$.*

Theorem 3. *Let a proper n-π-groupoid with marked n-tuple (a_1^n) be isotopic to a proper n-π-groupoid (G, B) with marked n-tuple (b_1^n), $A=B^T$, where $T=(\varphi_1^n, \psi)$. If (G, A_1) is an n-quasigroup which is partialy isotopic to (G, A), $A_1=A^R$, $R=(\alpha_1^n, a_1^n)$ and (G, B_1) is an n-quasigroup partially isotopic to (G, B), $B_1=B^S$, $S=(\beta_1^n, b_1^n)$ then the n-quasigroups A_1 and B_1 are isotopic, $A_1=B_1^T$.*

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