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THE LAPLACE TRANSFORM OF THE MODIFIED BESSEL FUNCTION OF THE SECOND KIND*

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Dedicated to the memory of Professor Arthur Erdélyi

1. Denoting the Mellin transform of $f \in L(0, \infty)$ by

$$\mathfrak{M}\{f(x): s\} = \int_0^\infty x^{s-1} f(x)\, dx, \tag{1}$$

we know that [2, Vol. I, p. 331, Eq. (26)]

$$\mathfrak{M}\{K_\nu(\alpha x): s\} = 2^{s-2} \alpha^{-s} \Gamma\left(\frac{1}{2} s - \frac{1}{2} \nu\right) \left(\frac{1}{2} s + \frac{1}{2} \nu\right), \tag{2}$$

provided that $\operatorname{Re}(\alpha) > 0$ and $\operatorname{Re}(s) > |\operatorname{Re}(\nu)|$.

If in the known result (2) we set $x = t^\rho$, $dx = \rho\, t^{\rho-1} dt$, ρ being a real number, and (for simplicity) replace s by s/ρ, $\rho \neq 0$, we shall at once have

$$\int_0^\infty t^{s-1} K_\nu(\alpha t^\rho)\, dt = \frac{2^{(s-2\rho)/\rho}}{|\rho|^{s/\rho}} \Gamma\left(\frac{s}{2\rho} - \frac{1}{2}\nu\right) \Gamma\left(\frac{s}{2\rho} + \frac{1}{2}\nu\right), \tag{3}$$

where, for convergence, $\operatorname{Re}(\alpha) > 0$, and *either*

$$\rho > 0, \ \operatorname{Re}(s/\rho) > |\operatorname{Re}(\nu)| \tag{4}$$

or

$$\rho < 0, \ \operatorname{Re}(\nu)| < \operatorname{Re}(s/\rho) < \frac{1}{2}. \tag{5}$$

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In order to compute the Laplace transform of the modified Bessel function $K_\nu(\alpha t)$, $\rho>0$, we put

$$(6)\qquad I_\rho^\mu(s)\equiv\int_0^\infty e^{-st}\,t^{\mu-1}\,K_\nu(\alpha\,t^\rho)\,dt,$$

expand the exponential function e^{-st} in powers of t, and integrate term-by-term using (3). We thus find that

$$(7)\qquad I_\rho^\mu(s)=\frac{2^{(\mu-2\rho)/\rho}}{\rho\alpha^{\mu/\rho}}\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\left(\frac{2}{\alpha}\right)^{n/\rho}\Gamma\left(\frac{\mu+n}{2\rho}-\frac{1}{2}\nu\right)\Gamma\left(\frac{\mu+n}{2\rho}+\frac{1}{2}\nu\right),$$

which holds true under the following sufficient) conditions:

$$(8)\qquad \rho>0,\ \operatorname{Re}(s)>0,\ \operatorname{Re}(\alpha)>0,\ \operatorname{Re}(\mu/p)>|\operatorname{Re}(\nu)|,$$

it being assumed that the series on the right makes sense.

Obviously, owing to the *second* inequality in (5), the integral (6) cannot be evaluated in this manner using (3) when $\rho<0$. To overcome this difficulty, however, we can appeal to the Eulerian integral for the Γ-function in conjunction with the (Mellin-Barnes) contour integral representation

$$(9)\qquad K_\nu(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}2^{s-2}\,\Gamma\left(\frac{1}{2}s-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}s+\frac{1}{2}\nu\right)x^{-s}\,ds,$$

$$x>0,\ \sigma>|\operatorname{Re}(\nu)|,$$

which indeed folows rather immediately from (2) and the Mellin inversion theorem. We thus obtain

$$(10)\qquad I_\rho^\mu(s)=\frac{s^{-\mu}}{4}\,H_{0,3}^{3,0}\left[\frac{\alpha}{2s^\rho}\,\middle|\,\overline{(\mu,\ -\rho),\left(\frac{1}{2}\nu,\ \frac{1}{2}\right),\ \left(-\frac{1}{2}\nu,\ \frac{1}{2}\right)}\right],$$

where $\rho<0$, $H_{p,q}^{m,n}[z|\ldots]$ denotes the familiar H-function of C. Fox ([3]), p. 408; see also [5], p. 214, Eq. (2.4) *et seq.*), and for convergence,

$$(11)\qquad \min\left\{\operatorname{Re}(s),\ \operatorname{Re}(\alpha),\ \operatorname{Re}\left(\mu-\frac{1}{2}\rho\right)\right\}>0.$$

Evidently, for $\mu=1$, our formulas (7) and (10) would provide the Laplace transforms of $K_\nu(\alpha t^\rho)$ for positive or negative values of ρ.

2. Special cases of $I_\rho^\mu(s)$

There are a number of interesting special situations in which the right -hand sides of our formulas (7) and (10) can be written in relatively more familiar forms. We, therefore, find it worthwhile to record the following special cases:

Case 1. When $\rho = m$, where m is a positive integer, we can make use of the elementary identity

$$(12)\qquad \sum_{n=0}^{\infty} \Phi(n) = \sum_{j=0}^{N-1} \sum_{n=0}^{\infty} \Phi(nN+j),\ N=1,\ 2,\ 3,\ \ldots,$$

which is easy to verify, and (7) would thus yield

$$(13)\qquad I_m^\mu(s) = \frac{2^{(\mu-2m)/m}}{m\,\alpha^{\mu/m}} \sum_{j=0}^{2m-1} \sum_{n=0}^{\infty} \frac{(-s)^{2mn+j}}{(2mn+j)!} \left(\frac{2}{\alpha}\right)^{2n+j/m} \cdot \Gamma\left(\frac{\mu+j}{2m} - \frac{1}{2}\nu + n\right) \Gamma\left(\frac{\mu+j}{2m} + \frac{1}{2}\nu + n\right),$$

provided that the conditions in (8) hold true with $\rho = m \in \{1, 2, 3, \ldots\}$.

By Gauss's multiplication theorem (*cf.* [1], Vol. I, p. 4, Eq. (11))], we have

$$(14)\qquad (2mn+j)! = j!\,(2m)^{2mn} \prod_{k=1}^{2m} \left(\frac{j+k}{2m}\right)_n,$$

where, for convenience, $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$, and (13) will readily reduce to the elegant form:

$$(15)\qquad I_m^\mu(s) = \frac{2^{(\mu-2m)/m}}{m\,\alpha^{\mu/m}} \sum_{j=0}^{2m-1} \frac{(-s)^j}{j!} \left(\frac{2}{\alpha}\right)^{j/m} \Gamma\left(\frac{\mu+j}{2m} - \frac{1}{2}\nu\right) \Gamma\left(\frac{\mu+j}{2m} + \frac{1}{2}\nu\right) \cdot {}_2F_{2m-1}\left[\begin{matrix} (\mu-m\nu+j)/2m,\ (\mu+m\nu+j)/2m; \\ (j+1)/2m,\ \ldots *\ \ldots,\ (j-2m)/2m; \end{matrix}\ \frac{4}{\alpha^2}\left(\frac{s}{2m}\right)^{2m}\right],$$

where m is a positive integer, $\mathrm{Re}(s)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\mu)>m\,|\mathrm{Re}(\nu)|$, and the asterisk in the generalzed hypergeometric functions repesents the fact that the denominator parameter $(2m)/2m$ is always omitted.

Remark 1. For $m=1$, the conditions $\mathrm{Re}(s)>0$ and $\mathrm{Re}(\alpha)>0$ can obviously be replaced by a single condition: $\mathrm{Re}(s+\alpha)>0$. Indeed, in this case our formula (15) would simplify considerably, and we first have

$$(16)\qquad I_1^{\mu}(s)=\frac{2^{\mu-2}}{\alpha^{\mu}}\left\{\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu\right){}_2F_1\left[\begin{matrix}\frac{1}{2}(\mu-\nu),\ \frac{1}{2}(\mu+\nu);\\ \frac{1}{2};\end{matrix}\ \frac{s^2}{\alpha^2}\right]\right.$$

$$\left.-\frac{2s}{\alpha}\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right){}_2F_1\left[\begin{matrix}\frac{1}{2}(\mu-\nu+1),\ \frac{1}{2}(\mu+\nu+1);\\ \frac{3}{2};\end{matrix}\ \frac{s^2}{\alpha^2}\right]\right\},$$

provided that $\mathrm{Re}(s+\alpha)>0$ and $\mathrm{Re}(\mu)>|\mathrm{Re}(\nu)|$.

Making use of Goursat's quadratic transformations (*cf.*, *e.g.*, [1], Vol. I, p. 111, Eq. (7) and Eq. (11)), the familiar (Euler's) formula [*op. cit.*, p. 64, (22)], and Legendre's duplication theorem [*op. cit.*, p. 5, Eq. (15)], we shall thus arrive at the well-known result [1, Vol. II, p. 50, Eq. (26)]:

$$(17)\qquad I_1^{\mu}(s)=\frac{(2\alpha)^{\nu}\sqrt{\pi}\,\Gamma(\mu-\nu)\,\Gamma(\mu+\nu)}{(s+\alpha)^{\mu+\nu}\,\Gamma\left(\mu+\frac{1}{2}\right)}\left[\begin{matrix}\mu+\nu,\ \nu+\frac{1}{2};\\ \mu+\frac{1}{2};\end{matrix}\ \frac{s-\alpha}{s+\alpha}\right],$$

where, for convergence, $\mathrm{Re}(s+\alpha)>0$ and $\mathrm{Re}(\mu)>|\mathrm{Re}(\nu)|$.

Case 2. When $\rho=-m$, where m is a positive integer, we can use Gauss's multiplication theorem once again to get the following special case of our formula (11):

$$(18)\qquad I_{-m}^{\mu}(s)=\frac{2^{\mu-m-1}\,m^{\mu-\frac{1}{2}}}{s^{\mu}\,\pi^{m-\frac{1}{2}}}\,G_{0,2m+2}^{2m+2,0}\left(\frac{\alpha^2}{4}\left(\frac{s}{2m}\right)^{2m}\,\middle|\,\overline{\Delta(2m;\mu),\ \frac{1}{2}\nu,\ -\frac{1}{2}\nu}\right),$$

where, as before, m is a positive integer, $\mathrm{Re}(s)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\mu)>-\frac{1}{2}m$, $G_{p,q}^{m,n}(z|\ldots)$ denotes the fimiliar (Meijer's) G-function (*cf.*, *e.g.*, [1], Vol. I, p. 207 *et seq.*), and for convenience, $\Delta(m;\lambda)$ denotes the set of m parameters λ/m, $(\lambda+1)/m$, ..., $(\lambda+m-1)/m$, $m\geq 1$.

Remark 2. For $\mu=1$, our formulas (15) and (18) would obviously yield the Laplace transforms of the modified Bessel functions $K_{\nu}(\alpha t^{\pm m})$, where $m=1, 2, 3, \ldots$; in fact, these consequences of our general results (7) and (11), and their *trivai* cases when $s=1$, can be shown to lead us fairly easily to the *corrected* (or modified) versions of the known results [4, p. 325, Eq. (1) and Eq. (2); p. 326, Eq. (3) end Eq. (4)], which were derived erroneously from certain integrals involving MacRobert's E-function (*cf.* [1], Vol. I, p. 203, § 5.2).

Case 3. Yet another interesting special case of our formula (7) would occur when $\rho=\frac{1}{2}$. Inded, in terms of the Whittaker functions [1, Vol. I, p. 264, Eq. (5)], we readily obtain the elegant formula (*cf.* [2]. Vol. I, p. 199, Eq. (37)):

$$(19)\qquad I^{\mu}_{\frac{1}{2}}(s)=\frac{\Gamma\left(\mu-\frac{1}{2}\nu\right)\Gamma\left(\mu+\frac{1}{2}\nu\right)}{\alpha s^{\mu-\frac{1}{2}}}\, e^{\alpha^2/8s}\, W_{\frac{1}{2}-\mu,\ \frac{1}{2}\nu}\left(\frac{\alpha^2}{4s}\right),$$

where, for convergence, $\mathrm{Re}(s)>0$, $\mathrm{Re}(\alpha)>0$, and $\mathrm{Re}(\mu)>\frac{1}{2}|\mathrm{Re}(\nu)|$.

Since [1, Vol. I, p. 265, Eq. (14)]

$$(20)\qquad W_{0,\nu}(z)=\sqrt{(z/\pi)}\,K_{\nu}\left(\frac{1}{2}z\right),$$

this last formula (19), with $\mu=\frac{1}{2}$, yields another known result (*cf.* [2], Vol. I, p. 199, Eq. (35)):

$$(21)\qquad I^{\frac{1}{2}}_{\frac{1}{2}}(s)=\frac{\sec\left(\frac{1}{2}\nu\pi\right)}{2\sqrt{(s/\pi)}}\, e^{\alpha^2/8s}\, K_{\frac{1}{2}\nu}\left(\frac{\alpha^2}{8s}\right),$$

provided that $\min\{\mathrm{Re}(s),\ \mathrm{Re}(\alpha)\}>0$ and $|\mathrm{Re}(\nu)|<1$.

Case 4. For $\rho=-\frac{1}{2}$, our formula (10) evidently gives us the result

$$(22)\qquad I^{\mu}_{-\frac{1}{2}}(s)=\frac{s^{-\mu}}{2}\, G^{3,0}_{0,3}\left(\frac{\alpha^2 s}{4}\,\middle|\,\begin{matrix} \\ \mu,\ \frac{1}{2}\nu,\ -\frac{1}{2}\nu\end{matrix}\right),$$

$$\min\left\{\mathrm{Re}(s),\ \mathrm{Re}(\alpha),\ \mathrm{Re}\left(\mu+\frac{1}{4}\right)\right\}>0,$$

in terms of the G-function occurring in (18).

Case 5. Our formula (7) simplifies also in the special case $\rho=m+\frac{1}{2}$, where m is a positive integer, the case $m=0$ being already covered by case 3. Applying the identities (12) and (14) appropriately, we thus find that

$$(23)\quad I^{\mu}_{m+\frac{1}{2}}(s)=\frac{4^{(\mu-2m-1)/(2m+1)}}{\left(m+\frac{1}{2}\right)\alpha^{(2\mu)/(2m+1)}}\sum_{j=0}^{2m}\frac{(-s)^j}{j!}\left(\frac{2}{\alpha}\right)^{2j/(2m+1)}$$

$$\cdot\Gamma\left(\frac{\mu+j}{2m+1}-\frac{1}{2}\nu\right)\Gamma\left(\frac{\mu+j}{2m+1}+\frac{1}{2}\nu\right)$$

$$\cdot\,{}_2F_{2m}\left[\begin{matrix}(j+1)/(2m+1)-\frac{1}{2}\nu,\ (\mu+j)/(2m+1)+\frac{1}{2}\nu; \\ (j+1)/(2m+1),\ldots *,\ldots,(j+2m+1)/(2m+1);\end{matrix}\ \frac{4}{\alpha^2}\left(\frac{s}{2m+1}\right)^{2m+1}\right],$$

where m is a positive integer, $\operatorname{Re}(s)>0$, $\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\mu)>\left(m+\frac{1}{2}\right)|\operatorname{Re}(\nu)|$, and, as before, the asterisk in the generalized hypergeometric functions represents the fact that the denominator parameter $(2m+1)/(2m+1)$ is always omitted.

Case 6. For $\rho=-m-\frac{1}{2}$, where m is a positive integer, our formula (10) would reduce to the form:

$$(24)\quad I^{\mu}_{-m-\frac{1}{2}}(s)=\frac{(2m+1)^{\mu-\frac{1}{2}}s^{-\mu}}{2^{m+1}\pi^m}$$

$$\cdot\,G^{2m+3,0}_{0,2m+3}\left(\frac{\alpha^2}{4}\left(\frac{s}{2m+1}\right)^{2m+1}\middle|\begin{matrix}\overline{} \\ \Delta(2m+1;\ \mu),\ \frac{1}{2}\nu,\ -\frac{1}{2}\nu\end{matrix}\right),$$

$$\min\left\{\operatorname{Re}(s),\ \operatorname{Re}(\alpha),\ \operatorname{Re}\left(2\mu+m+\frac{1}{2}\right\}>0,$$

which evidently would also incorporate Case 4 if we let $m=0$. {See also Equation (18) above.}

We remark in passing that such simplifications of our general results (7) and (10) as in Cases 5 and 6 above can be made whenever ρ is a *rational* number.

3. Further generalizations

In view of (9), if we define

$$\text{(25)} \qquad \Xi(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\prod_{j=1}^{q}\Gamma(\beta_j+B_j s)}{\prod_{j=1}^{r}\Gamma(\gamma_j-C_j s)\prod_{j=1}^{p}\Gamma(\alpha_j+A_j s)}x^{-s}\,ds,$$

where $x>0$, the A's, B's, and C's are positive numbers such that

$$\text{(26)} \qquad \Omega\equiv\sum_{j=1}^{p}A_j-\sum_{j=1}^{q}B_j-\sum_{j=1}^{r}C_j\leqq 0,$$

in which the equality holds provided that

$$\text{(27)} \qquad 0<x<R\equiv\prod_{j=1}^{p}(A_j)^{A_j}\prod_{j=1}^{q}(B_j)^{-B_j}\prod_{j=1}^{r}(C_j)^{-C_j},$$

and σ is any real number satisfying the inequality

$$\text{(28)} \qquad \sigma>\omega\equiv-\min_{1\leqq j\leqq q}\left\{\operatorname{Re}\left(\frac{\beta^j}{B^j}\right)\right\},$$

then it is easily varified that $\Xi(x)$ can be expressed as the sum of q Wright's generalized hypergeometric functions (*cf.* [6]). Furthermore, in terms of the familiar (Fox's) H-function [3, p. 408], we have the following relitionship (see also [5], p. 214, Eq. (2.4) *et seq.*)

$$\text{(29)} \qquad \Xi(z)=H_{p,\,q+r}^{q,\,0}\left[z\left|\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\\(\beta_1,B_1),\ldots,(\beta_q,B_q),(1-\gamma_1,C_1),\ldots,(1-\gamma_r,C_r)\end{matrix}\right.\right],$$

provided, for example, that

$$\text{(30)} \qquad |\arg(z)|<\frac{1}{2}\Lambda\pi,\quad \Lambda\equiv\sum_{j=1}^{q}B_j-\sum_{j=1}^{r}C_j-\sum_{j=1}^{p}A_j>0.$$

Equation (25), in conjunction with (1) and the Mellin inversion theorem, can be shown to imply

$$\text{(31)} \qquad \mathfrak{M}\{\Xi(x):s\}=\frac{\prod_{j=1}^{q}\Gamma(\beta_j+B_j s)}{\prod_{j=1}^{r}\Gamma(\gamma_j-C_j s)\prod_{j=1}^{p}\Gamma(\alpha_j+A_j s)},$$

where, for convergence, $\operatorname{Re}(s)>\omega$, ω being given by (28).

Evidently, in view of the relationship (29), this last formula (31) can also be deduced as a special case of our earlier result [5, p. 214, Eq. (2.14)].

Since the function $\Xi(z)$, defined by (25) under the conditions in (30), vanishes *exponentially* when $|z| \to \infty$ and $\Omega < 0$, we can appeal to (31) in order to derive a generalization of (7) given by

$$(32) \qquad \int_0^\infty e^{-st} t^{\mu-1} \Xi(\alpha t^\rho)\, dt = \rho^{-1} \alpha^{-\mu/\rho} \sum_{n=0}^{\infty} \frac{(-s\alpha^{-1/\rho})^n}{n!} \cdot \frac{\prod_{j=1}^{q} \Gamma[\beta_j + B_j(\mu+n)/\rho]}{\prod_{j=1}^{r} \Gamma[\gamma_j - C_j(\mu+n)/\rho] \prod_{j=1}^{p} \Gamma[\alpha_j + A_j(\mu+n)/\rho]},$$

which holds true whenever the series on the right has a meaning, $\rho > 0$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\alpha) > 0$, and $\mathrm{Re}(\mu) > \rho\omega$, ω being given by (28).

A similar generalization of our formula (10) can be obtained from the definition (25) by applying the method derivation of (10) *mutatis mutandis*, and we omit the details involved. Instead, however, we find it worthwhile to record an interesting unification of (7), (10) and (32), involving the general H-function, in the following form:

$$(33) \qquad \int_0^\infty e^{-ts} t^{\mu-1} H_{p,q}^{m,n}\left[\alpha t^\rho \,\middle|\, \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix}\right] dt$$

$$= s^{-\mu} H_{p+1,q}^{m,n+1}\left[\alpha s^{-\rho} \,\middle|\, \begin{matrix} (1-\mu, \rho), (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix}\right], \text{ if } \rho > 0,$$

$$= s^{-\mu} H_{p,q+1}^{m+1,n}\left[\alpha s^{-\rho} \,\middle|\, \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\mu, -\rho), (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix}\right], \text{ if } \rho < 0,$$

provided for example, that $\mathrm{Re}(s) > 0$, $|\arg(\alpha)| < \frac{1}{2}\Theta\pi$, where

$$(34) \qquad \Theta \equiv \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j > 0,$$

and *either*

$$(35) \qquad \mathrm{Re}(\mu) > -\rho \min_{1 \le j \le m} \left\{ \mathrm{Re}\left(\frac{\beta_j}{B_j}\right) \right\}, \text{ if } \rho > 0,$$

or

(36) $$\operatorname{Re}(\mu) > -\rho \max_{1 \leq j \leq n} \left\{ \operatorname{Re}\left(\frac{\alpha_j - 1}{A_j}\right) \right\}, \text{ if } \rho < 0.$$

Remark 3. Evidently, for $\mu = 1$, our last result (33) would provide us with the Laplace transform of the general $H_{p,q}^{m,n}[\alpha t^{\rho} | \ldots]$ function for real exponents $\rho \neq 0$, and under the appropriately modified conditions stated above. Furthermore, (33) can be rewritten in terms of the G-function whenever ρ, the *A's* and the *B's* are *rational* numbers.

We should like to conclude by remarking that a fairly wide variety of special functions, which occur rather frequently in problems of applied mathematics and mathematical analysis, can be expressed in terms of the H-function which evidently reduces, when $A_j = B_k = 1$, $j = 1, \ldots, p$; $k = 1, \ldots, q$, to the relatively more familiar G-function of Meijer (or the E-function of Mac Robert) and which includes, as a special case, the generalized hypergeometric ${}_p\psi_q[z]$ function of E. M. Wright (*cf.* [6]). Thus the Laplace transform pair, given by Equation (33), would aplly not only to the simpler special functions of mathematical physics such as the Bessel functions $J_\nu(z)$ and $I_\nu(z)$, the Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$, the Whittaker function $M_{k,\mu}(z)$, the Bessel-Wright function $J_\nu^\mu(z)$, the classical orthogonal polynomials of Hermite, Jacobi (and, of course, Gegenbauer, Legendre, and Tchebycheff), Laguerre, *et cetera*, which are particular cases of the generalized hypergeometric function ${}_pF_q[z]$ or ${}_p\psi_q[z]$, but also to the more involved Bessel functions $K_\nu(z)$ and $Y_\nu(z)$, the Whittaker function $W_{k,\mu}(z)$, their various combinations and several other related functions. Indeed, by making use of extensive tables of functions (and their various products and other combinations) expressibe in terms of the E- or G-function or the generalized hypergeometric functions (see, for instance, [1], Vol. I, pp. 215—222; [2], Vol. I, pp. 373—384; [2], Vol. II. pp. 433—444), the interested reader can easily construct from the results of this paper a considerably large tableau of higher transcendental functions and their respective images under the Laplace transform.

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