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ON EXISTENCE OF EXPANSION OF A COMPLEX FUNCTION

Miodrag Rašković

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The main result of this paper are theorems 1. and 2. which provide the existence of expansion of complex function, from vector subspace, under rather general conditions.

The proof is essentially relied on compactness theorem [1] and also the nonstandard complex (real) number idea [2].

Theorem 1. *Let V be a vector space and W its infinite dimensional subspace. Let $F(x, y, z, u, v)$ be a continuous function from C^5 in C, and f a function from W in C such that $F(f(\alpha x+\beta y), f(x), f(y), \alpha, \beta)=0$ and $f|(x)|\leqslant M$ for all $x, y\in W$ and $\alpha, \beta\in C$. Then f may be extended to whole V so that it satisfies the same conditions.*

Proof. Let $\mathscr{L}=\{+,\cdot, |\,|, \leqslant, \overline{C}(\,), \overline{V}(\,), \overline{R}(\,)\}\cup\{\overline{a}:a\in V\}\cup\{\overline{\alpha}:\alpha\in C\}$ be a language of the vector V over complex numbers set C.

Let us define a model for this langauge. The universum of that model is the set $V\cup C$.

Unary relations $\overline{R}(\,)$, $\overline{C}(\,)$, $\overline{V}(\,)$ are interpreted respectivly as the set of real numbers R, the set of complex numbers C and vector space V. The operation $+$ is usual addition in C and V, while for all $a\in V$ and $\alpha\in C$, $a+\alpha= =\alpha+a=0$. The operation $\cdot$ is general multiplication in C, multiplication by scalar in $C\times V$ (so that $\alpha a=a\alpha$) and trivial multiplication in V (that is, for all $x, y\in V$ $xy=0$, where 0 is zero in V). The norm is interpreted in C as complex numbers norm, and in V as trivial norm (that is, for every $x\in V$, $|x|=0_1$, where 0_1 is zero in C). Operations $+, \cdot, |\,|$, are introduced so that equation axioms are satisfied.

The relation $\leqslant$ is interpreted as the relation of order on R. Constants from $\{\overline{a}:a\in V\}\cup\{\overline{\alpha}:\alpha\in C\}$ are interpreted in natural way. Therefore $V_c= =(V\cup C, +, \cdot, |\,|, R(\,), C(\,), V(\,), \alpha, a)_{\substack{\alpha\in C\\ a\in V}}$ is the required model.

Let now $\mathscr{L}_{f,F}=\mathscr{L}\cup\{\bar{f}, \bar{F}\}$. Model for this language, for instance, is $W_c^{f,F}=(W_c, f, F)$ where f and F are functions which satisfy theorem conditions on W, and beyong W their value is zero. Let T be the set of following sentences:

(1) „elementary diagram for V"

(2) „elementary diagram for C"

(3) $(\forall x)(\bar{V}(x)\vee\bar{C}(x))$

(4) $(\forall x)(\bar{V}(x)\Rightarrow|\bar{f}(x)|\leqslant M)$

(5) $(\forall x)(\bar{V}(x)\Rightarrow(\exists y)(\bar{C}(y)\wedge\bar{f}(x)=y))$

(6) $\bar{f}(\bar{a})=\bar{\alpha}$ for all $a\in W$ and $\alpha\in C$ such that $W_c^{f,F}\models\bar{f}(\bar{a})=\bar{\alpha}$

(7) $(\forall a_1)(\forall a_2)(\forall\alpha_1)(\forall\alpha_2)(\bar{V}(a_1)\wedge\bar{V}(a_2)\wedge\bar{C}(\alpha_1)\wedge\bar{C}(\alpha_2)\Rightarrow$
$\Rightarrow\bar{F}(\bar{f}(\alpha_1 a_1+\alpha_2 a_2),\ \bar{f}(a_1),\ \bar{f}(a_2),\ \alpha_1,\ \alpha_2)=0$

(8) „function $\bar{F}$ is continuous"

(9) „$\bar{f}$ is a function from V to C"

Let us prove that each finite $T_1\subset T$ has a model. Let $\bar{a}_1,\ldots,\bar{a}_n$ be all constants from $\{\bar{a}:a\in V\}$ that appear in T_1 and $a_1,\ldots,a_n$ their interpretations. Let $E=\{e_\alpha:\alpha<\beta\}$ be some base of vector space V and $E_m=\{e_{k_1},\ldots,e_{k_m}\}$ its subbase through which the vectors $\{a_i:1\leqslant i\leqslant n\}$ are expressed. Let us denote by V_m finite dimensional subspace, generated by vectors from E_m. Now we choose subbase $E'_m=\{e'_{k_1},\ldots,e'_{k_m}\}$ in W, so that vectors from $E'_m\cap W$ repeat at the same place in sequence. Such a base exists because W is infinite dimensional. Let V'_m be the appropriate finite dimensional subspace of V. Let us define the function φ such that for $1\leqslant i\leqslant m$, $\varphi(e_{k_i})=e'_{k_i}$. Function φ can be extended to the function $\bar{\varphi}$ of subspace V_m to subspace V'_m, such that the fixed is on $V_m\cap W=V'_m\cap W$.

Obviously $V_{mc}^{f,F}=(V_m\cup C,\ +,\ \cdot,\ |\ |,\leqslant,\ C,\ R,\ V_m, f,\ F,\ \varphi(a_1),\ldots,\varphi(a_n),\ \alpha)$ $\alpha\in C$ is a model of the theory T_1.

As every finite subset of the theory T has a model, then, in accordance to the compactness theorem, the whole theory has a model. Let that model be ${}^*V_c^{*f,*F}$. After usual identification it could be taken that $V\subset{}^*V$ and $C\subset{}^*C$.

Taking (6) we get that ${}^*f\restriction W=f$ and taking (5) we get that for all $a\in V\setminus W$ exists $\alpha\in C$ such that ${}^*f(a)=\alpha$. Being that because of (4) $|\alpha|\leqslant M$, we can expand f from W to V, such that $f(a)=st{}^*f(a)=st\,\alpha$.

According to (7) we get that $^*F(^*f(\alpha_1 a_1+\alpha_2 a_2),\ ^*f(a_1),\ ^*f(a_2),\ \alpha_1,\alpha_2)=0$ for all $a_1,\ a_2\in V$ and $\alpha_1,\ \alpha_2\in C$. In accordance to (8) we have

$$0=st^*F(^*f(\alpha_1 a_1+\alpha_2 a_2),\ ^*f(a_1),\ ^*f(a_2),\ \alpha_1,\ \alpha_2)=$$

$$=F(st^*f(\alpha_1 a_1+\alpha_2 a_2),\ st^*f(a_1),\ st^*f(a_2),\ st\,\alpha_1,\ st\,\alpha_2)=$$

$$=F(f(\alpha_1 a_1+\alpha_2 a_2),\ f(a_1),\ f(a_1),\ f(a_2),\ \alpha_1,\ \alpha_2)$$

Obviously $V_c^{f,F}$ is a model of the theory T. The function $f:V\to C$ is the required extension.

Theorem 2. *Let V be a vector space, and W its infinite dimensional subspace. Let $F(x,\ y,\ z)$ be a continuous function from R^3 to R, and f a function from W to R such that $F(f(x+y),\ f(x),\ f(y))=0$ and $f|(x)|\leqslant M$ for all $x,\ y\in W$. Then f may be expanded to whole V so that it satisfies the same conditions.*

Proof. Analogous to the previous.

Example. Let be given the space c of all convergent sequences and its infinite dimensional subspace c_k of all sequences which are, in at most finite places diferent from zero.

Let us define the function f in c_k, such that for $(\xi_i)\in c_k$, $f((\xi_i))=$ $=\sin^2\left(\sum_{n=0}^{\infty}\xi_i\right)$.

The function f for every $x,\ y\in c_k$ satisfies condition $|f(x)|\leqslant 1$, and

$$(f(x+y)-f(x)(1-f(y))-f(y)(1-f(x)))^2=4f(x)f(y)(1-f(x))(1-f(y))$$

It is obvious that trivial or continuous extension is impossible. Esistence of the extension follows from theorem 2.

BIBLIOGRAPHY

[1] C. C. Chang and H. J. Keisler, *Model theory*, North-Holland, Amsterdam, 1973.

[2] K. D. Stroyan and W. A. J. Luxemburg, *Introduction to the theory of infinitesimals*, Academic Press, New York, 1976.