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ON QUASI-ALGEBRAS AND THE WORD PROBLEM

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1. Let Ω be a nonempty set whose membres we shall call *operation symbols*. We suppose that with each $\omega \in Q$ a non-negative integer $l(\omega)$ (the length of ω) is associated. Let $\Omega(n)$ denote the set of all ω with $l(\omega)=n$. Further, the members of the set

$$V=\{x_1, x_2, \ldots, x_n, \ldots\}$$

are called variables.

Denote by *Term* (Ω, V) the set[1] of all terms built up from the operation symbols and the variables.

By an algebraic law we understand any formula of the form

(1) $$t_1=t_2$$

where $t_1, t_2 \in$ *Term* (Ω, V).

Denote by L a set of certain algebraic laws (of the form (1)).

Further, let Γ be a given set[2] and let F be the free Ω-algebra generated by Γ and satisfying the laws L.

Let *Term* (Ω, Γ) denote the set[3] of all terms built up from Γ and Ω (i.e. without variables).

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[1] It is the smallest set which satisfies the following conditions

(i) $\Omega(0) \subset$ *Term* (Ω, V), $V \subset$ *Term* (Ω, V).

(ii) If $\omega \in \Omega(n)$ and $t_1, \ldots, t_n \in$ *Term* (Ω, V), then

$$\omega(t_1, \ldots, t_n) \in \textit{Term}(\Omega, V).$$

For the sets Ω, V we suppose that $\Omega \cap V = \varnothing$ and that the symbols), (, , are not members of $\Omega \cup V$.

[2] We suppose: $\Gamma \cap (\Omega \backslash \Omega(0)) = \varnothing$, $\Gamma \cup \Omega(0) \neq \varnothing$, $= \notin \Gamma$, $, \notin \Gamma$, $) \notin \Gamma$, $(\notin \Gamma$.

Elements of $\Gamma \cup \Omega(0)$ are called *constants*.

[3] If in the definition of *Term* (Ω, V) (see footnote 1) we put Γ instead of V, we obtain the definition of *Term* (Ω, Γ).

The elements of F are represented by equivalence classes of the following relation $\sim$ of the set $Term\,(\Omega, \Gamma)$:

(2) $t_1 \sim t_2$ *iff*: Term t_2 can be obtained from term t_1 by applying the laws L.

This definition is of any practical value only if we have an algorithm, i.e. a rule for deciding in a finite number of steps whether any two terms t_1, t_2 are equivalent. The problem of finding such an algorithm is called *the word problem* (for F).

Such an algorithm is decribed in the book *P. M. Cohn, Universal Algebra*, 1965. on the pages 155, 156.

We sketch it.

Let M be any subset of $Term\,(\Omega, \Gamma)$ satisfying the following condition:

(3) *For every* $t \in Term\,(\Omega, \Gamma)$ *there is at least one element* $m \in M$ *which is equivalent to* t.

Further, let σ: $Term\,(\Omega, \Gamma) \rightarrow M$ be a function satisfying the following conditions:

1° $\sigma(w) \sim \omega$

2° *If* $m \in M$, then $\sigma(m) = m$.

By means of M and σ we define an Ω-algebra F' as follows:

— The elements of F' are just the elements of M.

— Let $c \in \Omega(0)$ and $m \sim c$, where $m \in M$. With the symbol c a noughtary operation of M is associated, determined by the chosen element m.

— Let $\omega \in \Omega(n)$, where $n \geqslant 1$. With the symbol ω the following operation ω_M (of the set M) is associated

$$\omega_M(m_1, \ldots, m_n) = \sigma(\omega(m_1, \ldots, m_n))$$

According to the algorithm (see Theorem 9.1):

(4) If the algebra F' satisfies the laws L then the elements of M are pairwise nonequivalent, i.e.

$$m_1 = m_1 \Leftrightarrow m_1 \sim m_2 \qquad \text{(where } m_1, m_2 \in M\text{)}$$

However, that is not true as the following example shows.

Example 1. Let

$$\Omega = \{*\}, \ l(*) = 2, \ \Gamma = \{a\}, \ L = \{x_1 * x_1 = x_1\}^{4)}.$$

For the elements of M we choose:

$$a^2, \ a^3$$

where $a^2 \stackrel{\text{def}}{=} a * a$, $a^3 \stackrel{\text{def}}{=} a^2 * a$.

It is obvious that every term $t \in Term\,(\Omega, \Gamma)$ is equivalent to[5] a^2 or a^3.

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[4] We have written $x_1 * x_1$ instead of $*(x_1, x_1)$.

[5] As a matter of fact every term t is equivalent to the term a.

Further, let σ be the following function

$$\sigma=\begin{pmatrix} a^2 & a^3 & a^2*a^3 & a^3*a^3 \dots & t \dots \\ a^2 & a^3 & a^2 & a^3 \dots & a^2 \dots \end{pmatrix}$$

(t is any term different from a^2, a^3, a^2*a^2, a^3*a^3).

Then the algebra F' is a groupoid defined by the table:

$*_M$	a^2	a^3
a^2	a^2	a^2
a^3	a^2	a^3

The algebra F' satisfies the law $x_1*x_1=x_1$, but on the other hand the terms a^2, a^3 are equivalent.

One of our purposes is to correct the described algorithm.

2. Let Ω and Γ be given sets. So called *quasi-algebra* is a notion defined by them. The corresponding definition is a bit technically complex, therefore we begin with some examples of quasi-algebras.

Example 2. Let $\Omega=\{*\}$, $l(*)=2$, $\Gamma=\{a, b\}$. Let us denote by M the set of the following terms

$$a, \quad b, \quad a*b.$$

Further, let Q be the set of formulas:

$$\begin{array}{lll} a*a=b & a*b=a & a*(a*b)=a*b \\ b*a=a*b & b*b=a & b*(a*b)=a \\ (a*b)*a=a & (a*b)*b=a*b & (a*b)*(a*b)=b. \end{array} \tag{5}$$

Each of these formulas is of the form

$$t_1*t_2=t_3 \qquad \text{(where } t_1, t_2, t_3 \in M\text{)}.$$

The set Q is an example of a quasi-algebra, i.e. of a quasi-groupoid (of the set $\{a, b, a*b\}$).

More generally, if M is any subset of the set $Term(\Omega, \Gamma)$ containing the elements a and b, then each set M of formulas of the form

$$t_1*t_2=t_3 \qquad \text{(where } t_1, t_2, t_3 \in M;\ t_1, t_2 \text{ run over } M \text{ and for each pair } (t_1, t_2) \text{ there exists exactly one } t_3\text{)}$$

is an example of a quasi-algebra.

In order to work more easily we shall represent the quasi-algebra (5) by the following "table"

(5')

$*$	a	b	$a*b$
a	b	a	$a*b$
b	$a*b$	a	a
$a*b$	a	$a*b$	b

Similarly the table

(6)

$*$	a	b	$(a*a)*b$	$b*a$
a	b	a	$b*a$	b
b	$b*a$	$(a*a)*b$	b	$b*a$
$(a*a)*b$	a	b	$(a*a)*b$	a

represents a quasi-algebra, with

$$M=\{a, b, (a*a)*b, b*a\}.$$

We now return to the general case in order to define the notion of a quasi-algebra.

Let Ω, Γ be given sets and let M be any subset of *Term* (Ω, Γ) satisfying the following condition.

(7) If $t \in M$ and if $c \in \Omega(0) \cup \Gamma$ is any subterm of t, then $c \in M$.

Any quasi-algebra Q (related to the əsts Ω, Γ, M) is determined with

(8) $$Q=\bigcup_{\omega \in \Omega} Q_\omega$$

where Q_ω are certain sets of formulas, i.e. equalities.

If $l(\omega)=n \geqslant 1$, then Q_ω consists of certain equalities of the form

(9) $$\omega(t_1, \ldots, t_n)=t \qquad (t_1, \ldots, t_n, t \in M)$$

under the condition:

The terms $t_1, \ldots, t_n$ run over M, but two equalities of the form

$$\omega(t_1, \ldots, t_n)=t, \quad \omega(t_1, \ldots, t_n)=t' \qquad (t \neq t')$$

cannot belong to Q_ω.

If $l(\omega)=0$, then Q_ω consists of one equality of the form

(10) $$\omega=t$$

where $t \in M$.

Example 3. Let $\Omega=\{*, e, f, g\}$, $l(*)=2$, $l(e)=l(f)=l(g)=0$, $\Gamma=\{a\}$, $M=\{a, e, a*e\}$. The condition (7) is satisfied. Let Q_* be the set of equalities represented by the "table"

(11)

$*$	a	e	$a*e$
a	e	$a*e$	a
e	a	a	$a*e$
$a*e$	e	$a*e$	e

Further, let Q_e, Q_f, Q_g be the singletones with the members:

(12) $$e=e, \quad f=a*e, \quad g=e$$

respectively.

The set $Q_* \cup Q_e \cup Q_f \cup Q_g$ is an example of a quasi-algebra (related to the sets Ω, Γ, M).

3. We are going to study some general properties of quasi-algebras. At first we recall what means that an equality $t_1 = t_2$ is a consequence of given equalities.

Let E be a set of certain equalities containing some constants and operation symbols, but no variables.

We say that a given equality $t_1 = t_2$ is a consequence of E, in symbols $E \vdash t_1 = t_2$, if and only if there is a finite sequence of equalities

(13) $$u_1 = v_1, \quad u_2 = v_2, \ldots, u_n = v_n$$

such that u_n is t_1, v_n is t_2 and for every $i\,(1 \leqslant i \leqslant n)$ the equality $u_i = v_i$ satisfies:

1° $u_i = v_i$ is a member of E; *or*

2° $u_i = v_i$ is of the form $u = u$, i.e. u_i and v_i are the same term, *or*

3° $u_i = v_i$ can be obtained from certain of the equalities $u_1 = v_1$, $u_2 = v_2, \ldots, u_{i-1} = v_{i-1}$ by applying one of the following rules

$$\frac{p = q}{q = p}, \quad \frac{p = q,\ q = r}{p = r}, \quad \frac{p_1 = q_1, \ldots, p_k = q_k}{\omega(p_1, \ldots, p_k) = \omega(q_1, \ldots, q_k)}$$

where $l(\omega) = k$, and ω is an operation symbol occuring in some equality of E.

Now let Ω, Γ, L be given. Further, let

(14) $$L\,|_{Term(\Omega,\Gamma)}$$

be the set of all those equalities which can be obtained from the members of L i.e. from the equalities (1), by all posible replacements of the form

$$x_1 \to t_1, \quad x_2 \to t_2, \ldots, x_n \to t_n, \ldots$$

where $t_1, t_2, \ldots, t_n, \ldots$ run over the set Term (Ω, Γ).

Let Q be a quasi-algebra related to the sets Ω, Γ, M. We say that Q satisfies the laws L if and only if each member $t_1 = t_2$ of the set (14) is a consequence of Q.

Example. 4. Let $\Omega = \{*\}$, $l(*) = 2$, $\Gamma = \{a\}$, $M = \{a, a*a\}$ and Q a guasi-algebra defined by the table

(15)

$*$	a	$(a*a)*(a*a)$
a	$(a*a)*(a*a)$	$(a*a)*(a*a)$
$(a*a)*(a*a)$	a	a

Let the terms

$$(x_1 * x_2) * x_3, \quad x_1 * (x_2 * x_3)$$

be denoted by $L(x_1, x_2, x_3)$, $R(x_1, x_2, x_3)$ respectively.

From the given quasi-algebra Q it follows the equaliy

(16) $$L(a, a, a) = a.$$

One proof is:

(1′) $a*a=(a*a)*(a*a)$ (This equality is a member of Q)

(2′) $a=a$

(3′) $(a*a)*a=((a*a)*(a*a))*a$ (From (1′), (2′) by the rule $\dfrac{p_1=q_1,\ p_2=q_2}{p_1*p_2=q_1*q_2}$

(4′) $((a*a)*(a*a))*a=a$ (This equality is a member of Q)

(5′) $(a*a)*a=a$ (From (3′), (4′) by the rule $\dfrac{p=q,\ q=r}{p=r}$

i.e. $L(a, a, a)=a.$

Instead of such a detailed proof we can use the following:

$$
\begin{aligned}
L(a, a, a) &= (a*a)*a && \text{(Definition)}\\
&= ((a*a)*(a*a))*a && \text{(For } a*a=a*(a*a))\\
&= a
\end{aligned}
$$

Obviously, this proof corresponds to the usual procedure for finding the value of a given term.

Similarly, for the term $a*(a*a))$, say, we have

$$
\begin{aligned}
a*(a*(a*a)) &= a*(a*b) && \text{(From (15); } b \text{ is an abbrevation for } (a*a)*(a*a))\\
&= a*b && \text{(For } a*b=b)\\
&= b && \text{(For } a*b=b)
\end{aligned}
$$

Thus the equality

$$a*(a*(a*a))=(a*a)*(a*a)$$

follows from the quasi-algebra Q.

Generally, if $t(a, *)$ is any term, then at least one of the equalities

$$t(a, *)=a, \quad t(a, *)=(a*a)*(a*a)$$

follows from Q.

We now return again to the terms $L(x_1, x_2, x_3)$, $R(x_1, x_2, x_3)$.

We claim that the equality

$$L(x_1, x_2, x_3)=R(x_1, x_2, x_3)$$

i.e. the associative law is satisfied in the quasi-algebra Q.

First, let x_1, x_2, x_3 have the values a, a, a respectively. Then we have

$L(a, a, a)=a$ (Equality (16))

$R(a, a, a)=a*(a*a)=a*b=b$

i.e. $R(a, a, a)=(a*a)*(a*a).$

Does it mean that the quasi-algebra Q does not satisfy the associative law? However, from the quasi-algebra Q we also have the following argument

$R(a, a, a) = (a * a) * (a * a)$

$= b * b$ (For $a * a = b$, where b is an abbreviation for $(a * a) * (a * a)$)

$= a$ (From (15)

Thus, the equality

$$a = b, \quad \text{i.e.} \quad a = (a * a) * (a * a)$$

follows from the quasi-algebra Q. From this, we conclude that the set Q is equivalent[6] to the following quasi-algebra Q':

$*$	a
a	a

$\Omega = \{*\}, \quad \Gamma = \{a\}, \quad M' = \{a\}.$

Therefore the quasi-algebra Q obviously satisfies the associative law.

Let us return to the general study.

Let Q be a given quasi-algebra related to the set Ω, Γ, M, where, say

$$M = \{m_i \mid i \in I\}.$$

In general, members of M will be called *markers*.

Denote by *Term* (Q) the set of all terms built up from operation symbols of Ω and those members of Γ which occur in some marker.

Suppose that for a term $t \in$ *Term* (Q) and $m \in M$

$$Q \vdash t = m.$$

Then we say that m is a marker-value for the term t.

Using (7), (9) and (10) it is easy to prove[7] that

(17) *Each term $t \in$ Term (Q) has at least one marker-value.*

For some quasi-algebra Q it can happen that

(18) *Each term $t \in$ Term (Q) has exactly one marker-value.*

We say that a quasi-lgebra Q is *contractible* iff

(19) *There are two different*[8] *markers m_i, m_j such that $Q \vdash m_i = m_j$.*

It is easy to see that

[6] We say that two sets E_1, E_2 of equalities are equivalent iff each member of one of these sets is a consequance of the other set and vice versa.

[7] By induction on the number of all operation symbols ω $(l(\omega) \geqslant 1)$ occuring in the term t, say.

[8] *Different* as words, i.e. as terms.

(20) Each term $t \in Term\ (Q)$ has exactly one marker-value iff the quasi-algebra Q is not contractible.

For instance the quasi-algebra (15) is contractible (because the equality $a = (a * a) * (a * a)$ follows from Q).

Now we give examples of incontractible quasi-algebras.

Example 5. Let $\Omega = \{*, e, f, g\}$, $l(*) = 2$, $l(e) = l(f) = l(g) = 0$, $\Gamma = \{a, b\}$, $M = \{a, b, e, f\}$ and Q defined by the table

(21)

$*$	a	b	e	f
a	b	a	a	f
b	e	a	b	a
e	b	b	a	b
f	e	a	b	f

and the equalities

(22) $$e = e, \quad f = f, \quad g = f.$$

In that case:

(i) *Markers are some constants*, i.e. *some members of the set* $\Omega(0) \cup \Gamma$.

(ii) *The equalities of the type* (10) *have the form* $\omega = \omega$ *whenever* ω *is a marker.*

Just because of these reasons the equalities $(21) \cup (22)$ represent one part of a definition of an Ω-algebra Q'. Besides these equalities the definition of Q' still contains only the following formulas

(23) $$a \neq b, \quad a \neq e, \quad a \neq f$$
$$b \neq e, \quad b \neq f$$
$$e \neq f.$$

The Ω-algebra Q', defined by (21), (22), (23), is an Ω-algebra on the set $\{a, b, e, f\}$.

Generally we suppose that

(24) *Any Ω-algebra is incontractible*

Specially, from this it follows that the algebra Q' is incontractible, whence we conclude that the quasi-algebra Q is incontractible too.

The quasi-algebra Q is an example of so called *letter quasi-algebras*. These quasi-algebras are characterized by the preceding conditions (i), (ii). According to the general suposition (24) we have:

(25) *Any latter quasi-algebra Q is incontractible.*

We shall now describe a procedure for deciding whether a given quasi-algebra is contractible.

First we give an example.

Example 6. Let $\Omega=\{*, e, f\}$, $l(*)=2$, $l(e)=l(f)=0$, $\Gamma=\{a, b\}$, $M=\{a, b, e, (a*e)*b, (b*e)*(e*e)\}$ and Q representad by (26) and (27), where:

(26)

$*$	a	b	e	$(a*e)*b$	$(b*e)*(e*e)$
a	m_{11}	m_{12}	$(b*e)*(e*e)$	m_{14}	$(b*e)*(e*e)$
b	m_{21}	m_{22}	a	m_{24}	m_{25}
e	m_{31}	m_{32}	$(b*e)*(e*e)$	m_{34}	m_{34}
$(a*e)*b$	m_{41}	m_{42}	m_{43}	m_{44}	m_{45}
$(b*e)*(e*e)$	m_{51}	$(a*e)*b$	m_{53}	m_{54}	m_{55}

(27) $$e=e, \quad f=(a*e)*b.$$

For the m_{ij} it is supposed that they may be any elements of M.

The markers $(a*e)*b$, $(b*e)*(e*e)$ are called *composite*, because they contain the sign $*$. Generally, we say that a marker m (of a certain quasi-algebra) is *composite* iff m contains at least one $\omega\in\Omega$, where $l(\omega)\geqslant 1$.

In the *first* step with each composite marker we associate one new symbol-its *new denotation*. Let 4, 5, say, be new denotations for the markers

$$(a*e)*b, \quad (b*e)*(e*e)$$

respectively.

According to this we form the following equalities

(28) $$4=(a*e)*b, \quad 5=(b*e)*(e*e).$$

This is the *second* step of our procedure.

In the *third* step, in the equalities of a given quasi-algebra Q we replace all composite markers by their new denotations. So we obtain the equalities (26′), (27′), where[9]:

(26′)

$*$	a	b	e	4	5
a	m'_{11}	m'_{12}	5	m'_{14}	5
b	m'_{21}	m'_{22}	a	m'_{24}	m'_{25}
e	m'_{31}	m'_{32}	5	m'_{34}	m'_{35}
4	m'_{41}	m'_{42}	m'_{43}	m'_{44}	m'_{45}
5	m'_{51}	4	m'_{53}	m'_{54}	m'_{55}

(27′) $$e=e, \quad f=4.$$

Any use of new denotations cannot be creative. From this, investigation whether the given quasi-algebra $(25)\cup(27)$ is conctractible may be replaced by investigation whether the set $(26')\cup(27')\cup 28)$ is contractible (i.e whether from the equalities $(26')\cup(27')\cup(28)$ any of the following equalities

$$\begin{array}{rrrr} a=b, & a=e, & a=4, & a=5 \\ & b=e, & b=4, & b=5 \\ & & e=4, & e=5 \\ & & & 4=5 \end{array}$$

follows).

[9] We suppose that m'_{ij} is obtained from m_{ij}.

The *fourth* step is investigation of contractibility of the set $(26') \cup (27') \cup (28)$.

Let us denote the set $(26') \cup (27')$ by Q'. This set may be considered as a letter quasi-algebra, with:

$$\Omega = \{*, e, f\}, \quad l(*) = 2, \quad l(e) = l(f) = 0$$

$$\Gamma = \{a, b, 4, 5\}, \quad M = \{a, b, e, 4, 5\}.$$

From this and (25) we conclude that Q' is incontractible.

Now we cheek if the equalities (28) follow from Q'.

For that purpose from Q' we "calculate" the composite markers $(e * e) * b$, $(b * e) * (e * e)$. So we have:

$$\begin{aligned} (a * e) * b &= 5 * b && (\text{For } a * e = 5) \\ &= 4 && (\text{For } 5 * b = 4) \\ (b * e) * (e * e) &= a * 5 && (\text{For } b * e = a, \quad e * e = 5). \\ &= 5 \end{aligned}$$

The previons calculations constitute the *fifth* step, which is also the last one.

Thus the equalities (28) follow from Q'.

From this we conclude that the set $(26') \cup (27') \cup (28)$ is equivalent to the set Q', which is incontractible.

Finally, we conclude that the quasi-algebra Q is incontractible.

Note, that the result of investigation essentially depends on the calculations done in the fifth step. Namely:

(29) *The quasi-algebra Q is incontractible if and only if the values of composite markers, calculated from the quasi-algebra Q', are equal to their new denotation.*

For instance, suppose that we have obtained the equality

$$(a * e) * b = a.$$

Then from this equality and the equality $4 = (a * e) * b$ it should follow $a = 4$, i.e. the set $(26') \cup (27') \cup (28)$ should be contractible.

In general case we proceede quite similarly as in the given example. The main steps are:

— *introduction new denotations for composite terms*

— *calculation of values of all composite term (related to the new quasi--algebra Q').*

In general case we use the equivalence (29), *by means of which we finally decide about contractibility of the given quasi-algebra Q.*

We have seen that for any letter quasi-algebra Q (with $M = \{m_i \mid i \in I\}$) there is exactly one Ω-algebra Q' with elements a_i satisfying the equalities defining the quasi-algebra Q. The Ω-algebra Q' is determined by the equalities which define Q *plus* the inequalities of the following type

$$a_i \neq a_j \qquad (\text{where } i \neq j).$$

The similar holds for any incontractible quasi-algebra Q too. Namely if Q is any incontractibile quasi-algebra related to the given sets Ω, Γ, $M = m_i \mid i \in I\}$ defined by the equalities (9) and (10), then there exists exactly one Ω-algebra Q' with elements $m_i \in M$ and satisfying (9) and (10). This algebra Q' is determined by (9), (10) and the inequalities of the following type

$$(30) \qquad m_i \neq m_j \qquad (\text{where } i \neq j$$

This follows from the fact that the set (9)$\cup$(10)$\cup$(30) is consistent (since Q is incontractible). For instance, if Q is the quasi-algebra considered in Example 6, then corresponding Ω-algebra Q' has as elements the terms[10]:

$$a,\ b,\ e,\ (a*e)*b,\quad (b*e)*(e*e)$$

the operation $*$ is defined by (26) and the noughtary operations e, f are defined by (27).

4. Now we return to the word problem posed at the beginning. If the sets Ω (the set of some operation symbols), L (the set of some algebraic laws) and Γ are given then, as we have already seen, the main problem is to describe all members of the corresponding free algebra F. These members are equivalence classes of the relation $\sim$ defined by (2). More precisely this definition reads:

$$(31) \qquad t_1 \sim t_2 \quad \textit{iff} \quad L \mid_{Term(\Omega,\Gamma)} \vdash t_1 = t_2$$

where t_1, t_2 run over the set *Term* (Ω, Γ).

As at the beginning we suppose that there exists a subset M of the set *Term* (Ω, Γ) satisfying (3).

In addition, for the set M we suppose[11] (7).

Let $\omega \in \Omega$ (with $l(\omega) = n \geqslant 1$) and $m_1, \ldots, m_n \in H$.

From (3) it follows that the term $\omega(m_1, \ldots, m_n)$ is equivalent to at least one element $m \in M$ i.e. to at least one *marker* (as before for the elements of M we shall use the name *marker*). According to this let us form the equality

$$(32) \qquad \omega(m_1, \ldots, m_n) = m$$

where m is a chosen marker equivalent to $\omega(m_1, \ldots, m_n)$. Of course this choise need not be unique.

Further, let $\omega \in \Omega(0)$ and let $m \in M$ be a chosen marker equivalent to ω. Let us form the equality:

$$(33) \qquad \omega = m.$$

10) As elements we can also use

$$a,\ b,\ e.\ 4,\ 5,$$

11) As a matter of fact from (3) it can be easily proved that there exists a new set M_1 satisfuing both (3) and (7).

We denote by Q the set of all such formed equalities (32) and (33). The set Q may be considered a quasi-algebra related to the set Ω, Γ, M. For Q we shall also say that it is

a marker quasi-algebra related to Ω, Γ, L.

Our main result is the following theorem.

Theorem. *The markers are pairwise nonequivalent if and only if the marker quasi-algebra* Q

1° *is incontractible, and*

2° *satisfies the laws* L.

Proof. From the definition of Q it follows:

$$L\,|_{\,Term\,(\Omega,\Gamma)} \vdash Q. \tag{34}$$

Using (34) we conclude that the set $L\,|_{\,Term\,(\Omega,\Gamma)}$ is equivalent to the set $L\,|_{\,Term\,(\Omega,\Gamma)} \cup Q$. From this we conclude that the relation defined by (31), can be defined by:

$$m_1 \sim m_2 \quad \textit{iff} \quad L\,|_{\,Term\,(\Omega,\Gamma)} \cup Q \vdash m_1 = m_2. \tag{35}$$

We first suppose that the conditions 1° and 2° hold. As Q satisfies the laws L the set $L\,|_{\,Term\,(\Omega,\Gamma)} \cup Q$ is equivalent to the set Q. From this we conclude that definition (35) may be replaced by:

$$m_1 \sim m_2 \quad \textit{iff} \quad Q \vdash m_1 = m_2.$$

At last, from the condition 1° it follows:

$$m_1 \sim m_2 \quad \textit{iff} \quad m_1, m_2 \text{ are the same term.}$$

Now we suppose that at least one of the conditions 1°, 2° does not hold.

Case: 1° *does not hold.* Then for some two different markers t_1, t_2 we have

$$M \vdash t_1 = t_2.$$

From this and (35) follows that $t_1 \sim t_2$.

Case: 2° *does not hold.* Then for some different markers m_1, m_2 we have

$$L\,|_{\,Term\,(\Omega,\Gamma)} \cup Q \vdash m_1 = m_2 \tag{36}$$

because for at least one law $u = v$ (a member of L) the coresponding marker Q-values[12] of the left and right side will be different, m_1, m_2 say.

From (35) and (36) it follows that $m_1 \sim m_2$. Thus in both casses 1°, 2° there are two different markers m_1, m_2 which are equivalent. The proof is completed.

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[12] i.e. related to the marker quasi-algebra Q.

Remark. The condition (3), (5), 1° and 2° are natural because they correspond to the following properties of the free algebra F:

F is an Ω-algebra, generated by $\Omega(0) \cup \Gamma$ and satisfying the laws L.

At the end we give two examples.

Example 7. Let $\Omega = \{*\}$, $l(*) = 2$, let Γ be any non empty set and let the associative law

$$(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3) \tag{37}$$

be the only member of L. For markers we can take all terms of the form:

$$p_1, p_1 * p_2, \quad (p_1 * p_2) * p_3, \ldots$$

i.e. of the form $\prod_{i=1}^{n} p_i$, where $n = 1, 2, \ldots$ and $p_i \in \Gamma$.

A coressponding marker quasi-algebra Q can be defined by the equalities of the form

$$\prod_{i=1}^{n} p_i * \prod_{i=n+1}^{n+m} p_i = \prod_{i=1}^{n+m} p_i. \tag{38}$$

It is easy to see that Q is incontractible and satisfies (37).

For instance, let us compute $(p_1 * p_2) * p_3$ using (38). So we have:

value of $(p_1 * p_2) * p_3$

$= $ *value of* ((*value of* $p_1 * p_2) * p_3$)

$= $ *value of* $(A * p_3)$ $\quad A$ is an abbrevation for $p_1 * p_2$

$= A * p_3$

$= (p_1 * p_2) * p_3$.

Generally, it is true:

value of $\prod_{i=1}^{n} p_i$, *related to* Q, *equals* $\prod_{i=1}^{n} p_i$

i.e. Q is incontractible[13].

Example 8. Let us construct the group G with the presentation

$$G = Gp\{a, b \mid a^2 = b^2 = e, \quad a \cdot b = b \cdot a\}.$$

This problem may be considered as the problem of finding the free algebra F with: $\Omega = \{\cdot, e, {}^{-1}\}$. $(l(\cdot) = 2,\ l(e) = 0,\ l({}^{-1}) = 1)$

$\Gamma = \{a, b\}$, L is the set of the following equalities:

$$x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3, \quad x_1 \cdot x_1^{-1} = e, \quad x_1 \cdot e = x_1 \tag{39}$$

$$a \cdot a = e, \quad b \cdot b = e, \quad a \cdot b = b \cdot a. \tag{40}$$

It is easy to se that as markers we can use:

$$a, \quad a, \quad b, \quad a \cdot b. \tag{41}$$

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[13] As we can see introduction of new denotations for composite terms is not necessary.

Further, a corresponding quasi-algebra Q is determined by the following equalities

(42)

$*$	e	a	b	$a\cdot b$
e	e	a	b	$a\cdot b$
a	a	e	$a\cdot b$	b
b	b	$a\cdot b$	e	a
$a\cdot b$	$a\cdot b$	b	a	e

$e^{-1}=e,\quad a^{-1}=a,\quad b^{-1}=b,\quad (a\cdot b)^{-1}=a:b$

$e=e$

The only composite marker is $a\cdot b$. For it we have

value of $a\cdot b=a\cdot b$ (because in (42) we have the equality $a\cdot b=a\cdot b$).

It can be also proved that (42) satisfies the laws (39) and (40).
Thus the terms (41), represent all members of the asked group.