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ON OSTROWSKI'S FUNDAMENTAL EXISTENCE THEOREM IN THE COMPLEX CASE

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Abstract. In [4] we have given another proof of Ostrowski's fundamental existence theorem for Newton-Raphson method in the real case and, as generalization of those ideas, we have proved the Theorem 2 which, in the case of real polynamials, gives sufficient conditons for convergence of Newton-Raphson method in terms of initial point only. In this paper we prove the corresponding theorem in the case of complex polynomials. The indicated theorem is the following:

Theorem. *Let*

$$f(z)=z^n+p_{n-1}z^{n-1}+\cdots+p_1 z+p_0$$

be a complex polynomial, $f(z_0)\cdot f'(z_0)\neq 0$, $h_0=-\dfrac{f(z_0)}{f'(z_0)}$, $z_1=z_0+h_0$, $K_0=\{z\in C \mid |z-z_1|\leqslant|h_0|\}$. *Form, starting with* z_0, *the sequence* (z_i) *by the recurrence formula*

$$z_{i+1}=z_i-\frac{f(z_i)}{f'(z_i)} \qquad (i=0,\ 1,\ldots)$$

and assume that z_0 *satisfies the condition*

$$M(z_0)\cdot|h_0|\leqslant q\cdot|f'(z_0)| \qquad (q\approx 0{,}2313463410)$$

where $M(z)=\max\{|f'(z)|,\ |f''(z)|,\ldots,\ |f^{(n)}(z)|\}-z$ *is a fixed complex number. Then all* z_i *lie in* K_0 *and we have*

$$z_i\to\zeta \qquad (i\to\infty)$$

where ζ *is the unique zero of* f *in the circle* K_0. *Unless it lies on the boundary of* K_0, ζ *is a simple zero. Further, we have the relations*

(a) $$|z_{i+1}-z_i|\leq\frac{e^q-1-q}{q^2}\,\frac{M(z_i)}{2\,|f'(z_i)|} \qquad (i=1,\ 2,\ldots)$$

(b) $$|\zeta-z_{i+1}|\leq\frac{e^q-1-q}{q^2}\,\frac{M(z_i)}{2\,|f'(z_i)|}\,|z_i-z_{i+1}| \qquad (i=1,\ 2,\ldots)$$

where $(e^q-1-q):q^2\approx 0{,}5408950673$, The proof of the theorem, with the exception of one point — the proof of the uniqueness, is as in the real case [4]. To prove that in the complex case ζ is the only zero in the circle K_0, we must proceed differently than in the real case.

First of all we introduce some definitions:

$$h_i \overset{\text{def}}{=\!=} \frac{-f(z_i)}{f'(z_i)}, \text{ i.e. } h_i = z_{i+1} - z_i \qquad (i=0,\ 1, \ldots)$$

$$K_i \overset{\text{def}}{=\!=} \{z \in C \mid |z - z_{i+1}| \leq |z_{i+1} - z_i|\} \qquad (i=0,\ 1, \ldots)$$

Supose now that ζ^* is a zero of $f(z)$ in K_0, i.e.

(1) $$f(\zeta^*) = 0, \qquad |\zeta^* - z_1| \leq |h_0|$$

Therefrom and from the identity

$$f(z) = f(z_i) + \frac{f'(z_i)}{1!}(z - z_i) + \frac{f''(z_i)}{2!}(z - z_i)^2 + \cdots + \frac{f^{(n)}(z_i)}{n!}(z - z_i)^n$$

we obtain

(2) $$f(z_i) = -\frac{f'(z_i)}{1!}(\zeta^* - z_i) - \frac{f''(z_i)}{2!}(\xi^* - z_i)^2 - \cdots - \frac{f^{(n)}(z_i)}{n!}(\zeta^* - z_i)^n$$

Further, by identity

$$f'(z_i)(\zeta^* - z_{i+1}) = f'(z_i)(\zeta^* - z_i) - f'(z_i)(z_{i+1} - z_i)$$

and by definition of sequence (z_i) we have

(3) $$f'(z_i)(\zeta^* - z_{i+1}) = -\frac{f''(z_i)}{2!}(\zeta^* - z_i)^2 - \cdots - \frac{f^{(n)}(z_i)}{n!}(\zeta^* - z_i)^n$$

Therefore it follows immediately:

$$|f'(z_i)|\,|\zeta^* - z_{i+1}| \leq M(z_i)\,|\zeta^* - z_i|^2 \left[\frac{1}{2!} + \frac{1}{3!}|\zeta^* - z_i| + \cdots + \frac{1}{n!}|\zeta^* - z_i|^{n-2}\right]$$

$$\leq M(z_i)\,|\zeta^* - z_i|^2 \left[\frac{1}{2!} + \frac{1}{3!}2|h_0| + \cdots + \frac{1}{n!}(2|h_0|)^{n-2}\right]$$

(While for z inside K_i we have $|\zeta^* - z_i| \leq 2|h_i|$)

$$\leq M(z_i)\,|\zeta^* - z_i|^2 \left[\frac{1}{2!} + \frac{1}{3!}2q + \cdots + \frac{1}{n!}(2q)^{n-2}\right]$$

(While $|h_0| \leq q$)

$$\leq \frac{1}{(2q)^2} M(z_i)\,|\zeta^* - z_i|^2 \left[\frac{1}{2!}(2q)^2 + \frac{1}{3!}(2q)^3 + \cdots\right]$$

$$= \frac{e^{2q} - 1 - 2q}{(2q)^2} M(z_i)\,|\zeta^* - z_i|^2$$

Thus, we have just proved the inequality

$$|f'(z_i)|\,|\zeta^*-z_{i+1}|\leq\frac{e^{2q}-1-2q}{(2q)^2}M(z_i)\,|\zeta^*-z_i|^2 \tag{4}$$

which is equivalent to

$$\frac{(2q)^2}{e^{2q}-1-2q}\,\frac{|\zeta^*-z_{i+1}|}{|\zeta^*-z_i|}\leq\frac{|\zeta^*-z_i|}{|f'(z_i)|}M(z_i) \tag{5}$$

We introduce

$$g_i\overset{\text{def}}{=}\frac{(2q)^2}{e^{2q}-1-2q}\cdot\frac{|\zeta^*-z_{i+1}|}{|\zeta^*-z_i|} \tag{6}$$

Then

$$g_i\leq\frac{|\zeta^*-z_i|}{|f'(z_i)|}M(z_i) \tag{7}$$

We have[1] from (6)

$$\prod_{i=0}^{n-1}g_i=\left[\frac{(2q)^2}{e^{2q}-1-2q}\right]^n\cdot\frac{|\zeta^*-z_n|}{|\zeta^*-z_0|} \tag{8}$$

and therefrom

$$\prod_{i=0}^{n-1}g_i\,\frac{|\zeta^*-z_0|}{|h_0|}\cdot\frac{|h_0|\cdot M(z_0)}{|f'(z_n)|}\left[\frac{e^{2q}-1-2q}{(2q)^2}\right]^n\frac{|\zeta^*-z_n|}{|f'(z_n)|}\cdot M(z_0)$$

Using the inequality (16) proved in [4] we easily obtain

$$M(z_n)\leq e^{nq}M(z_0). \tag{9}$$

Therefore, by (7)

$$g_n\leq\left(\prod_{i=0}^{n}g_i\right)\frac{|\zeta^*-z_0|}{|h_0|}\,\frac{|h_0|\,M(z_0)}{|f'(z_n)|}\left[\frac{e^q(e^{2q}-1-2q)}{(2q)^2}\right]^n. \tag{10}$$

In [4] it has been proved

$$|f'(z_{i+1})|\geqslant(2-e^q)\,|f'(z_i)|.$$

Implying this inequulity n times we obtain

$$|f'(z_n)|\geqslant(2-e^q)^n\,|f'(z_0)|$$

and from (10)

$$g_n\leq\left(\prod_{i=0}^{n-1}g_i\right)\frac{|\zeta^*-z_0|}{|h_0|}\,\frac{|h_0|\,M(z_0)}{|f'(z_0)|}\alpha^n$$

$$\left(\text{The expression }\frac{e^q(e^{2q}-1-2q}{(2-e^q)(2q)^2}\text{ is designed by }\alpha\right)$$

$$\leqslant\left(\prod_{i=0}^{n-1}g_i\right)\frac{2|h_0|}{|h_0|}g\,\alpha^n$$

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1) For $n=0$ the product $\prod_{i=0}^{n-1}g_i$ is, by definition, *one*.

$$\left(\text{While } |\zeta^* - z_0| \leq 2|h_0|, \ \frac{|h_0| M(z_0)}{|f'(z_0)|} \leq q\right)$$

$$= \left(\prod_{i=0}^{n-1} g_i\right) \cdot 2 q \alpha^n$$

Thus, we have just proved

$$g_n \leq \left(\prod_{i=0}^{n-1} g_i\right) \cdot 2 q \alpha^n \tag{11}$$

Using the inequalities2)

$$q \leq 1/2, \qquad \alpha \leq 1$$

we conclude

$$g_n \leq \prod_{i=0}^{n-1} g_i \tag{12}$$

For $n=0$, the right-hand side of (12) is equal to *one*. Hence $q_0 \leq 1$ and by induction we see that generally $g_n \leq 1$. From (6) we now have

$$|\zeta^* - z_{i+1}| \leq \frac{e^{2q} - 1 - 2q}{(2q)^2} |\zeta^* - z_i|.$$

While for $q \approx 2313463410$ the inequality

$$\frac{e^{2q} - 1 - 2q}{(2q)^2} < 1$$

holds, we conclude

$$z_i \to \zeta^* \qquad (i \to \infty)$$

and therefore $\zeta^* = \zeta$ which completes the proof of the uniqueness in the complex case.

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2) The value $q \approx 0{,}2313463410$ has just been so determined that the inequality $\alpha \leq 1$ holds.