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GLOBAL AND LOCAL SATURATION THEOREMS IN SOME SPACES OF TEMPERATE FUNCTIONS

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Introduction

In this paper we consider the saturation properties for a family of operators of convolution type

$$K_\rho f(x)=\int_{-\infty}^{\infty} f(x-t)\,k_\rho(t)\,dt=(f*k_\rho)(x) \tag{1}$$

where f is a real locally integrable function on R and, for a given $k\in L^1$, $k_\rho(x)=\rho\,k(\rho\,x)$ (ρ is a positive parameter tending to infinity). The family of operators (1) is usually called a singular integral and the function k kernel (of the singular integral).

The (global) saturation theorem for singular integrals (1) is well known in the spaces L^p, $1\leqslant p<\infty$, [3]. The object of the present paper is to obtain generalizations of this result in two directions.

First, in Section 2 we prove the global saturation theorem in some spaces of temperate functions (Definition 1.1), which are larger than the L^p spaces, and then in Section 3 we obtain the corresponding local result.

As we are dealing with convolution integrals, the main tool in the proofs is the Fourier transform method. However, even in the spaces L^p, $p>2$, considerable difficulties arise from the fact that the Fourier transform cannot be defined as a function.

Several authors [2, 5, 6] have developed different methods for surmounting these difficulties (in the spaces L^p, $1\leqslant p<\infty$). We shall present another slightly different method, which seems to be appropriate even in the spaces of temperate functions and for global as well as for local results.

1. Temperate functions

In this section we prove some preliminary results on products and convolutions of some special temperate distributions. In all the distribution theoretic notions we follow L. Schwartz [8] except that infinitely differentiable functions on R will be shortly called smooth and functions which are bounded by $|x|^{-r}$, $x\to\infty$, for all $r>0$, rapidly decreasing at infinity. Thus, $\mathcal{D}(a, b)$ will be the space of smooth functions with compact support in (a, b) and $\mathcal{S}$ the space of smooth rapidly decreasing functions, equipped with the usual topologies. $\mathcal{S}'$, the dual of $\mathcal{S}$, is the space of temperate distributions.

We shall be particularly concerned with the regular distributions from $\mathcal{S}'$ (i.e. distributions which can be identified with locally integrable functions) and we shall call such distributions temperate functions.

Definition 1.1. Let $\alpha>0$. F_α will be the space of all locally integrable functions f such that

$$\|f\|_\alpha=\int\limits_{-\infty}^{\infty}\frac{|f(x)|}{(1+x^2)^{\alpha/2}}dx<\infty. \tag{2}$$

Obviously, F_α under the norm (2) is a Banach space; $\alpha<\beta$ implies that $F_\alpha\subset F_\beta$; $F_\alpha\subset\mathcal{S}'$ for every $\alpha>0$, moreover $\bigcup\limits_{\alpha>0}F_\alpha=F$, the space of all temperate functions from $\mathcal{S}'$.

As already mentioned, the main tool in the proofs will be the Fourier transform method. In particular, we shall make use of the following well known property: The Fourier transform of the convolution product is the (pointwise) product of the transforms

$$(f*k)^\wedge=\hat{f}\cdot\hat{k}. \tag{3}$$

The interpretation of equation (3), which is immediate in the case when $f\in L^p$, $1\leqslant p\leqslant 2$, $k\in L^1$, is not as easy if we let f be an arbitrary temperate distribution. Indeed, it is well known [8] that in this case k has to satisfy a rather restrictive condition: $(1+x^2)^{r/2}k$ has to be bounded, for every $r>0$. This condition in turn yields the smoothness of the Fourier transform $\hat{k}$.

On the other hand, the Fourier transform method leads to the following characterization of a function f belonging to the saturation class

$$\hat{g}=|x|^\alpha\hat{f} \tag{4}$$

(where g is chosen appropriately).

We are led to the same problem as before (3): temperate distributions can, in general, be multiplied only by smooth functions, and $|x|^\alpha$ or $\hat{k}(x)$ are obviously not smooth.

This problem was solved, for the spaces L^p, by E. Görlich [5] who replaced $|x|^\alpha$ in (4) by the smooth function $(1+x^2)^{\alpha/2}$ having the same behaviour at infinity (and thus yielding the same saturation class), or by J. Boman [2] who proved that functions from L^p may be convolved with temperate distributions which are Fourier transforms of homogeneous functions ($|x|^\alpha$).

We shall proceed somewhat differently and prove that a temperate function f may be convolved with distributions having a prescribed (finite) rate of decrease at infinity (Proposition 1.1) or, which is essentially the same, that its Fourier transform $\hat{f}$ may be multiplied by functions having only a finite number of derivatives (Proposition 1.4). This will enable us to treat both (3) and (4) in a similar way (Corollaries 1.1 and 1.2).

Before proceeding, we cite the following characterization of bounded distributions from [8]. A bounded distribution is defined as a continuous linear functional on the space $\mathcal{D}_{L^1}$, which consists of smooth functions belonging together with all their derivatives to L^1.

Lemma 1.1. [8, II, p. 95]. *Let T be a temperate distribution.. Then $(1+x^2)^{\alpha/2}T$ is a bounded distribution for some $\alpha>0$, if and only if $(1+x^2)^{\alpha/2}(T*\varphi)(x)$ is a bounded function, for all $\varphi\in\mathcal{S}$.*

Proposition 1.1. *Let $\alpha>0$. Let $T\in\mathcal{S}'$ be such that $(1+x^2)^{\alpha/2}T$ is a bounded distribution. Then the convolution $f*T\in\mathcal{S}'$ is well defined, for every $f\in F_\alpha$.*

Proof. For $\varphi\in\mathcal{S}$ it follows by the definition of convolution [8]

$$\langle T*f,\varphi\rangle=\langle T_x,\langle f_t,\varphi(x+t)\rangle\rangle=\langle(1+x^2)^{\alpha/2}T_x,\frac{1}{(1+x^2)^{\alpha/2}}\langle f_t,\varphi(x+t)\rangle\rangle.$$

Since $(1+x^2)^{\alpha/2}T$ is a bounded distribution, all we have to prove is that $(1+x^2)^{-\alpha/2}\langle f_t,\varphi(x+t)\rangle\in\mathcal{D}_{L^1}$. But this is easily established using the fact that $f\in F_\alpha$.

Proposition 1.2. *Let $f\in F_\alpha$, $\alpha>0$, and let ψ be a temperate function such that $(1+x^2)^{\alpha/2}\hat{\psi}$ is a bounded distribution. Then the product $\psi\cdot\hat{f}$ is well defined as a temperate distribution.*

Proof. Note that if ψ is smooth then ψ satisfies the assumption of the Proposition (moreover $(1+x^2)^{\alpha/2}\hat{\psi}$ is a bounded distribution for every $\alpha>0$). In this case the product $\psi\cdot\hat{f}$ exists [8] and for every $\varphi\in\mathcal{S}$

$$\langle\psi\cdot\hat{f},\varphi\rangle=\langle\hat{f},\psi\varphi\rangle=\langle f,(\psi\varphi)^\wedge\rangle=\int_{-\infty}^{\infty}f(x)(\psi\varphi)^\wedge(x)\,dx. \tag{5}$$

On the other hand, even if ψ is not smooth, but only satisfies the assumptions of the Proposition, we can show that the integral in (5) is absolutely convergent and then define the product by the following equation

$$\langle \psi \cdot \hat{f}, \varphi \rangle = \int_{-\infty}^{\infty} f(x) (\psi\varphi)\hat{}\,(x)\, dx, \quad \varphi \in \mathcal{S}. \tag{6}$$

To this end we first notice that

$$(\psi\varphi)\hat{}\,(x) = (\hat{\psi} * \hat{\varphi})(x) \tag{7}$$

and that from the fact $(1+x^2)^{\alpha/2}\hat{\psi}$ is a bounded distribution it follows by Lemma 1.1

$$\sup_{x\in R} (1+x^2)^{\alpha/2} |(\hat{\psi} * \hat{\varphi})(x)| \leqslant C. \tag{8}$$

Thus we have

$$\left| \int_{-\infty}^{\infty} f(x) (\psi\varphi)\hat{}\,(x)\, dx \right| \leqslant \sup_{x\in R} (1+x^2)^{\alpha/2} |(\psi\varphi)\hat{}\,(x)| \int_{-\infty}^{\infty} \frac{|f(x)|}{(1+x^2)^{\alpha/2}} dx$$

$$= \sup_{x\in R} (1+x^2)^{\alpha/2} |(\hat{\psi} * \hat{\varphi})(x)| \, \|f\|_\alpha \leqslant C_1$$

by (7) and (8), establishing (6). We may proceed in a similar fashion to show that this product is separately continuous.

P r o p o s i t i o n 1.3. *Let* $\alpha>0$ *and let* f *and* ψ *satisfy the assumptions of Proposition* 1.2. *Then*

$$(\hat{\psi} * f)\hat{} = \psi \cdot \hat{f}. \tag{9}$$

P r o o f. According to Propositions 1.1 and 1.2 both sides in (9) are well defined. We thus have, for every $\varphi \in \mathcal{S}$

$$\langle (\hat{\psi} * f)\hat{}, \varphi \rangle = \langle \hat{\psi} * f, \hat{\varphi} \rangle = \langle f_x, \langle \hat{\psi}_t, \hat{\varphi}(x+t) \rangle \rangle = \langle f, \hat{\psi} * \hat{\varphi} \rangle =$$

$$= \langle f, (\psi\varphi)\hat{} \rangle = \langle \psi \cdot \hat{f}, \varphi \rangle$$

by the definition of Fourier transform, convolution and product.

C o r o l l a r y 1.1. *Let* $k \in L^1$ *be such that* $(1+x)^{\alpha/2} |k(x)| \leqslant C$, *for some* $\alpha>1$. *Then, according to Proposition* 1.1 *the convolution* $k*f$ *is well defined, for* $f \in F_\alpha$, *and according to Proposition* 1.3

$$(k*f)\hat{} = \hat{k} \cdot \hat{f}.$$

Moreover, it is easily checked that $k*f \in F_\alpha$ ([1]) and, if we set $k_\rho(x) = \rho k(\rho x)$, that $k_\rho * f$ tends to f in F_α, $\rho \to \infty$, which means that the operator (1) defines an approximation process in $F_\alpha \cdot (\alpha>1$ is necessary for the validity of the last assertion).

Further let us consider the function $\psi_\alpha(x)=|x|^\alpha$, $\alpha>0$. The Fourier transform of this function is the temperate distribution given by the regularisation of the following function

$$\frac{|x|^{-\alpha-1}}{2\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}} \tag{10}$$

([4], p. 173). Denote the distribution (10) by χ_α, that is to say $\chi_\alpha=\hat{\psi}_\alpha$. It is easily seen that (10) is meaningful for $\alpha>0$ and is indeed an entire function of α [4].

Corollary 1.2. *Let $\psi_\alpha(x)=|x|^\alpha$, $\alpha>0$ and $\chi_\alpha=\hat{\psi}_\alpha$. Then for $f\in F_{\alpha+1}$ it follows*

$$(f*\chi_\alpha)\hat{}=\psi_\alpha\cdot\hat{f}. \tag{11}$$

Indeed it is readily seen from (10) that $(1+x^2)^{-\frac{\alpha+1}{2}}\chi_\alpha$ is a bounded distribution so that Proposition 1.3 may be applied to get (11).

Definition 1.2. Let χ_α be defined by (10) and let $f\in F_{\alpha+1}$. The distribution $f\mathbin{\dot{*}}\chi_\alpha\in\mathcal{S}'$ is called the α th Riesz derivative of f and is denoted by $f^{\{\alpha\}}$ or $D^{\{\alpha\}}f$.

From (11) it follows that $g\in\mathcal{S}'$ is the α th Riesz derivative of the function $f\in F_{\alpha+1}$ if and only if

$$\hat{g}=|x|^\alpha\hat{f}. \tag{12}$$

As already mentioned, relations of this type play a central role in characterizations of the saturation class and this is the reason Riesz derivative has to be introduced (cf. [3]) instead of the more usual Riemann-Liouville (for which the corresponding „Fourier transform equation" is $\hat{g}=(\mathrm{i}v)^\alpha\hat{f}$). However, in contrast to the Riemann-Liouville derivative, which is generated by the kernel $x_+^{-\alpha-1}$ supported in $(0,\infty)$ and thus may be applied to arbitrary temperate distributions (supported in $(0,\infty)$), the Riesz derivative, generated by the kernel $|x|^{-\alpha-1}$ (with an unbounded support), can be applied only to distributions having a prescribed order of growth at infinity. This produces the following discrepancy; for a function $f\in F_\alpha$, however „good" its α th Riesz derivative may be, the βth derivative, $\beta<\alpha$, cannot be defined by $f*\chi_\beta$, $(1+x^2)^{-\frac{\alpha+1}{2}}\chi_\beta$ being unbounded. Still, this inconveniance is a result of the behaviour at infinity of the function f and not of its smoothess properties, so that it can be improved by considering the local Riesz derivatives (cf. Section 3).

Further we shall have to consider measures from $\mathcal{S}'$.

Definition 1.2. Let $\alpha>0$. M_α will be the space of all functions g of locally bounded variation on R such that

$$\|g\|_{M_\alpha}=\int_{-\infty}^{\infty}\frac{|dg(x)|}{(1+x^2)^{\alpha/2}}<\infty.$$

Obviously, every measure from $\mathcal{S}'$ may be looked at as an element of $\bigcup_{\alpha>0} M_\alpha$. Besides, $F_\alpha\subset M_\alpha$ (if we identify those measures $g\in M_\alpha$ which are absolutely continuous with their derivatives). Let us note the following fact, of which we shall make constant use: M_α is the conjugate Banach space of

$$C_\alpha=\{\varphi\in C(R)\mid \lim_{x\to\infty}(1+x^2)^{\alpha/2}\varphi(x)=0\} \tag{13}$$

with the norm

$$\|\varphi\|_{C_\alpha}=\sup_{x\in R}(1+x^2)^{\alpha/2}|\varphi(x)|.$$

We are now in position to prove the following

Proposition 1.4. *Let the α th Riesz derivative of the temperate function ψ be a temperate measure. Then, for $f\in F_\alpha$, the product $\psi\cdot\hat{f}$ is well defined in $\mathcal{S}'$.*

Proof. According to Proposition 1.2 it is enough to show that $(1+x^2)^{\alpha/2}\hat{\psi}$ is a bounded distribution.

To this end let us first prove the following simple statement. If T is a temperate measure, then its Fourier transform $\hat{T}$ is a bounded distribution.

Indeed, for $u\in\mathcal{D}_{L^1}$ we have by definition

$$\langle\hat{T},u\rangle=\langle T,\hat{u}\rangle=\int_{-\infty}^{\infty}\hat{u}(x)\,dT(x). \tag{14}$$

Now, for $u\in\mathcal{D}_{L^1}$, $\hat{u}$ is a continuous rapidly decreasing function and T being an element of M_β (for some $\beta>0$) it follows that the integral in (14) is convergent, whence it is readily deduced, by using standard arguments, that $\hat{T}$ is a bounded distribution.

Turning to the proof of the Proposition, let the temperate measure μ be the α th Riezs derivative of ψ, i.e.

$$\hat{\mu}=|v|^\alpha\cdot\hat{\psi},$$

by (12). It follows now from the preceding that $|v|^\alpha\hat{\psi}$ is a bounded distribution, and this is easily proved to be equivalent with the boundedness of the distribution $(1+x^2)^{\alpha/2}\hat{\psi}$, which completes the proof of the Proposition.

We finally formulate several simple propositions which essentially assert that smooth functions from F_α have Riesz derivatives.

Lemma 1.2. *Let $\varphi\in\mathcal{S}$. Then for every $\alpha>0$ the α th Riesz derivative is the continuous function given by*

$$\varphi^{\{\alpha\}}(x)=\int_{-\infty}^{\infty}|v|^{\alpha}\hat{\varphi}(v)\,e^{ivx}\,dv.$$

Furthermore $\varphi^{\{\alpha\}}$ satisfies the condition $(1+x^2)^{\frac{\alpha+1}{2}}|\varphi^{\{\alpha\}}(x)|\leqslant C$.

The first assertion of the Lemma follows from (12), and the second by Lemma 1.1, in view of the definition of Riesz derivative $\varphi^{\{\alpha\}}=\varphi*\chi_\alpha$.

Lemma 1.3. *Let $f\in F_{\alpha+1}$, $\alpha>0$ and $\varphi\in\mathcal{S}$. Then $f*\varphi$ has an α th Riesz derivative in $F_{\alpha+1}$ and*

$$(f*\varphi)^{\{\alpha\}}(x)=(f*\varphi^{\{\alpha\}})(x). \tag{15}$$

Indeed, since by Lemma 1.2 and Corollary 1.1 $f*\varphi\in F_{\alpha+1}$ the assertion follows by applying the Fourier transfom to (15).

Lemma 1.4. *Let $f, g\in F_{\alpha+1}$, $\alpha>0$. Then*

$$\int_{-\infty}^{\infty} f(x)\,\varphi^{\{\alpha\}}(x)\,dx=\int_{-\infty}^{\infty}\varphi(x)\,g(x)\,dx,\ \forall\,\varphi\in\mathcal{S}$$

if and only if

$$\hat{g}=|x|^{\alpha}\hat{f}.$$

Since by Corollary 1.2 the product $|x|^{\alpha}\hat{f}$ is well defined, the assertion follows by evaluation of the Fourier transform of g.

2. Global saturation theorem

In this section we prove a saturation theorem for singular integrals (1) in the spaces F_α. Conditions (*S*) and (*M*) in Theorem 2.1 are the standard ones required for (global) saturation in the spaces L^p [3]. The condition $(1+x^2)^{\frac{\alpha+1}{2}}|k(x)|\leqslant C$ makes the operator applicable to the space $F_{\alpha+1}$ and is, as we shall see later, necessary for local saturation even in the spaces L^p.

The elements of the saturation class will be characterized by the fact that their Riesz derivatives are temperate measures in the very same way the saturation class in L^1 was characterized by the existence of *BV* Riesz derivatives.

Theorem 2.1. *Let $\alpha>0$. Let $f\in F_\beta$, for some β, $1<\beta\leqslant\alpha+1$, and let the kernel k of the singular integral* (1) *satisfy the following conditions*

(S) $\displaystyle\lim_{v\to 0}\frac{\hat{k}(v)-1}{|v|^{\alpha}}=c>0,$

(M) $\displaystyle\exists\,\lambda\in L^1\ \hat{\lambda}(v)=\frac{\hat{k}(v)-1}{|v|^{\alpha}},$

(B) $(1+x^2)^{\beta/2}|k(x)|\leqslant C,\quad (1+x^2)^{\beta/2}|\lambda(x)|\leqslant C.$

Then

(i) $\|f-K_\rho f\|_\beta o=(\rho^{-\alpha})$, $\rho\to\infty$ $\Rightarrow f$ *is a polynomial of degree* $<\beta-1$,

(ii) $\|f-K_\rho f\|_\beta=0(\rho^{-\alpha})$, $\rho\to\infty$ $\Leftrightarrow f$ *has an* α *th Riesz derivative in* M_β.

Remark. A similar statement is valid in the spaces F_β, $0<\beta\leqslant 1$, except that β in condition (B) has to be replaced by $\beta+1$ (otherwise the singular integral (1) might even be non convergent in F_β — *cf.* Corollary 1.1). But since we are mostly interested in operators which satisfy (B) with $\beta=\alpha+1$ (Section 3), this restriction is not very important. Besides, instead of F_β, $0<\beta\leqslant 1$, we usually consider their subspaces L^p, $1\leqslant p<\infty$, for which condition (B) is indeed superfluous.

In the proof of the Theorem we shall have need of the following lemma.

Lemma 2.1. *Let* f *and* k *satisfy the hypotheses of Theorem* 2.1 *and let* $\varphi\in\mathcal{S}$. *Then*

$$\lim_{\rho\to\infty}\int_{-\infty}^{\infty} f(x)\,(\rho^\alpha(K_\rho\varphi(x)-\varphi(x))-\varphi^{\{\alpha\}}(x))\,dx=0. \tag{16}$$

Proof. We first notice that for every ρ

$$\sup_{x\in R}(1+x^2)^{\beta/2}\,|\,\rho^\alpha(K_\rho\varphi(x)-\varphi(x))-\varphi^{\{\alpha\}}(x)\,|\leqslant C. \tag{17}$$

Indeed, since $K_\rho\varphi(x)=(\varphi*k_\rho)(x)$ and $\varphi^{\{\alpha\}}(x)=(\varphi\mathbin{\dot*}\chi_\alpha)(x)$, (17) follows by Lemma 1.1. From (17) it follows that the integral in (16) is absolutely convergent, in view of $f\in F_\beta$.

For a given $\varepsilon>0$, take $M>0$ such that

$$\int_{|x|>M}|f(x)\,[\rho^\alpha(K_\rho\varphi(x)-\varphi(x))-\varphi^{\{\alpha\}}(x)]\,|\,dx<\frac{\varepsilon}{2}. \tag{18}$$

For this M we have

$$\int_{-M}^{M}|f(x)\,[\rho^\alpha(K_\rho\varphi(x)-\varphi(x))-\varphi^{\{\alpha\}}(x)]\,|\,dx\leqslant$$

$$\leqslant\sup_{|x|<M}(1+x^2)^{\beta/2}\,|\,\rho^\alpha(K_\rho\varphi(x)-\varphi(x))-\varphi^{\{\alpha\}}(x)\,|\int_{-\infty}^{\infty}\frac{|f(x)|}{(1+x^2)^{\beta/2}}\,dx$$

$$\leqslant\|f\|_\beta(1+M^2)^{\beta/2}\sup_{|x|<M}|\,\rho^\alpha(K_\rho\varphi(x)-\varphi(x))-\varphi^{\{\alpha\}}(x)\,|<\frac{\varepsilon}{2} \tag{19}$$

if ρ is large enough. Indeed, it is well known that, when the kernel k of the singular integral (K_ρ) satisfies the conditions (S) and (M), we have for every $\varphi\in\mathcal{S}$

$$\sup_{x\in R}|\,\rho^\alpha(K_\rho\varphi(x)-\varphi(x))-\varphi^{\{\alpha\}}(x)\,|\to 0,\quad \rho\to\infty.$$

Thus (16) follows by adding (18) and (19), which completes the proof of the Lemma.

Proof of the Theorem. First of all observe that for f and k satisfying the assumptions of the Theorem we have

$$\int_{-\infty}^{\infty} f(x)\, K_\rho \varphi(x)\, dx = \int_{-\infty}^{\infty} K_\rho f(x)\, \varphi(x)\, dx, \quad \forall \varphi \in \mathcal{S}$$

so that by Lemma 2.1. we obtain

$$\lim_{\rho\to\infty} \rho^\alpha \int_{-\infty}^{\infty} (K_\rho f(x) - f(x))\, \varphi(x)\, dx = \int_{-\infty}^{\infty} \varphi^{\{\alpha\}}(x) f(x)\, dx. \tag{20}$$

(i) From the assumption $\rho^\alpha \| K_\rho f - f \|_\beta = o(1)$, $\rho \to \infty$, the weak convergence follows, thus

$$\lim_{\rho\to\infty} \rho^\alpha \int_{-\infty}^{\infty} (K_\rho f(x) - f(x))\, \varphi(x)\, dx = 0, \quad \forall \varphi \in \mathcal{S}.$$

Compairing with (20) we get

$$\int_{-\infty}^{\infty} \varphi^{\{\alpha\}}(x) f(x)\, dx = 0, \quad \forall \varphi \in \mathcal{S},$$

which by Lemma 1.4. means $|v|^\alpha \hat{f} = 0$. It follows that supp $\hat{f} \subset \{0\}$, whence f is a polynomial [8]. Since $f \in F_\beta$, we conclude that the degree of the polynomial f is less than $\beta - 1$.

(ii) Assume $\rho^\alpha \| K_\rho f - f \|_\beta = 0(1)$, $\rho \to \infty$. It follows that the family $\rho^\alpha (K_\rho f - f)$ is bounded in the space M_β, which is the conjugate Banach space for C_β (13). The weak* compactness theorem now yields the existence of a measure $g \in M_\beta$ and a sequence (ρ_j) with $\lim \rho_j = \infty$, $j \to \infty$, such that

$$\lim_{j\to\infty} \rho_j^\alpha \int_{-\infty}^{\infty} (K_{\rho_j} f(x) - f(x))\, \varphi(x)\, dx = \int_{-\infty}^{\infty} \varphi(x)\, dg(x) \tag{21}$$

holds for every $\varphi \in \mathcal{S}$. Thus by (20) and (21) we have

$$\int_{-\infty}^{\infty} f(x)\, \varphi^{\{\alpha\}}(x)\, dx = \int_{-\infty}^{\infty} \varphi(x)\, dg(x)$$

which by Lemma 1.4 means that

$$\hat{g} = |v|^\alpha \hat{f}. \tag{22}$$

Thus $g \in M_\beta$ is the α th Riesz derivative of $f \in F_\beta$, proving the „inverse" part of the saturation theorem.

Conversely, assume that there exists a temperate measure g such that (22) holds. Since by (B) both k and λ satisfy the assumptions of Corollary 1.1 we have

$$\rho^\alpha (K_\rho f - f)\,\hat{} = \rho^\alpha (\hat{k}_\rho(v) - 1)\hat{f} = \frac{\hat{k}(v/\rho) - 1}{(|v|^\alpha/\rho^\alpha)} |v|^\alpha \hat{f} =$$

$$= \hat{\lambda}(v/\rho)\,\hat{g} = \hat{\lambda}_\rho \cdot \hat{g} = (\lambda_\rho * dg)\,\hat{}$$

According to the uniqueness theorem for the Fourier transform it follows

$$\rho^\alpha (K_\rho f(x) - f(x)) = (\lambda_\rho * dg)(x). \tag{23}$$

On the other hand, $g \in M_\beta$ and λ satisfying condition (B) implies $\lambda_\rho * dg \in F_\beta$ and moreover

$$\|\lambda_\rho * dg\| F_\beta = 0(1),$$

It thus follows from (23) that

$$\rho^\alpha \|K_\rho f - f\|_\beta = 0(1)$$

which completes the proof of the theorem.

Corollary 2.1. (Characterization of the saturation class)

Let $\alpha > 0$ and let $f \in F_\beta$, $1 < \beta \leqslant \alpha + 1$. Then f has an α th Riesz derivative in M_β if and only if

$$\|R_\varepsilon^\alpha f\|_\beta = \left\| \frac{1}{L_\alpha} \int_\varepsilon^\infty \frac{\overline{\Delta}_u^{2m} f}{u^{1+\alpha}} du \right\|_\beta = 0(1), \quad \varepsilon \to 0$$

where $m \in N$, $2m > \alpha$, $\overline{\Delta}_u^{2m} f(x)$ is the $2m$th Riemann central difference of f,

$L_\alpha = (-1)^m 2^{2m-\alpha} \int_0^\infty u^{-1-\alpha} \sin^{2m} u\, du$ ([3], p. 409).

Sketch of the proof. The operator R_ε^α can be written in the from $R_\varepsilon^\alpha = \varepsilon^{-\alpha}(K_\varepsilon^\alpha - I)$, where K_ε^α is a singular integral, whose kernel $k_\varepsilon^\alpha(x) = \varepsilon^{-1} k(x/\varepsilon)$ satisfies the conditions of Theorem 2.1. This assertion is proved in [3] for $0 < \alpha < 2$, and for $\alpha \geqslant 2$ can be proved similarly. Thus Theorem 2.1 can be applied to conclude that f has an α th Riesz derivative in M_β if and only if $\|R_\varepsilon^\alpha f\|_\beta = 0(1)$, $\varepsilon \to 0$.

Moreover, it follows from the above argument that $f \in F_\beta$ belongs to the saturation class of an operator satisfying the assumptions of Theorem 2.1 if and only if $\|R_\varepsilon^\alpha f\|_\beta = 0(1)$, $\varepsilon \to 0$.

3. Local saturation theorem

In this section we establish the local analogues of the results of the preceding section. Obviously, when we replace in (2) the infinite interval of integration by a finite interval (a, b), all the norms $\|\cdot\|_\alpha$ become equivalent to $L(a, b)$. Thus the local saturation theorem in F_α has the following form (we first prove the inverse part).

Theorem 3.1. *Let $\alpha>0$ and let $f\in F_{\alpha+1}$. Let the kernel k of the singular integral* (1) *satisfy the conditions* (*S*) *and* (*M*) *and*

$$\text{(A)} \qquad (1+x^2)^{\frac{\alpha+1}{2}}\,|k(x)|\leqslant C.$$

Then

$$\text{(i)}\ \ \|K_\rho f-f\|_{L(a,b)}=o(\rho^{-\alpha}),\ \ \rho\to\infty\ \Rightarrow\ \forall\varphi\in\mathcal{D}(a, b)\ \int_{-\infty}^{\infty} f(x)\,\varphi^{[\alpha]}(x)\,dx=0,$$

$$\text{(ii)}\ \ \|K_\rho f-f\|_{L(a,b)}=O(\rho^{-\alpha}),\ \ \rho\to\infty\ \Rightarrow$$

$$\Rightarrow \exists g\in BV(a, b)\,\forall\varphi\in\mathcal{D}(a, b)\ \int_{-\infty}^{\infty} f(x)\,\varphi^{[\alpha]}(x)\,dx=\int_{-\infty}^{\infty}\varphi(x)\,dg(x).$$

Proof. Let $\varphi\in\mathcal{D}(a, b)$. In the same way equation (20) was obtained in the proof of Theorem 2.1 we get

$$\text{(24)} \qquad \lim_{\rho\to\infty}\rho^\alpha\int_{-\infty}^{\infty}(K_\rho f(x)-f(x))\,\varphi(x)\,dx=\int_{-\infty}^{\infty}\varphi^{[\alpha]}(x)\,f(x)\,dx.$$

(i) Since $\rho^\alpha\|K_\rho f-f\|_{L(a,b)}=o(1)$, $\rho\to\infty$, it follows that for every $\varphi\in\mathcal{D}(a, b)$

$$\lim_{\rho\to\infty}\rho^\alpha\int_a^b(K_\rho f(x)-f(x))\,\varphi(x)\,dx=0$$

so that by (24) we get

$$\int_{-\infty}^{\infty} f(x)\,\varphi^{[\alpha]}(x)\,dx=0,\quad \forall\varphi\in\mathcal{D}(a, b).$$

(ii) The assumption $\rho^\alpha\|K_\rho f-f\|_{L(a,b)}=O(1)$, $\rho\to\infty$, and the weak* compactness of the space $BV(a, b)$ yield the existence of a function $g\in BV(a, b)$ and a sequence (ρ_j) with $\lim\limits_{j\to\infty}\rho_j=\infty$ such that

$$\lim_{j\to\infty}\rho_j^\alpha\int_a^b(K_{\rho_j}f(x)-f(x))\,\varphi(x)\,dx=\int_a^b\varphi(x)\,dg(x),\quad\forall\varphi\in\mathcal{D}(a, b).$$

Compairing with (24) we obtain

$$\int_{-\infty}^{\infty} f(x)\,\varphi^{\{\alpha\}}(x)\,dx = \int_a^b \varphi(x)\,dg(x), \quad \forall\, \varphi \in \mathcal{D}(a, b).$$

Theorem 3.2. *Let f and k be as in Theorem 3.1. Then*

(i) $\| K_\rho f - f \|_{L(a,b)} = o(\rho^{-\alpha}),\ \rho \to \infty \Rightarrow$ *f is a polynomial of degree $2m > \alpha$ in (a, b).*

(ii) $\| K_\rho f - f \|_{L(a,b)} = 0(\rho^{-\alpha}), \quad \rho \to \infty \quad \Rightarrow \quad \| R_\varepsilon^\alpha f \|_{L(a',b')} = 0(1), \quad \varepsilon \to 0$ *where $a < a' < b' < b$.*

Proof. (ii) Let $h \in \mathcal{D}(-1, 1)$, $h_n(x) = n \cdot h(nx)$, $n \in N$ and set $f_n(x) = (f * h_n)(x)$. Then for n large enough

$$\| K_\rho f_n - f_n \|_{L(a',b')} = \Big\| \int_{-1/n}^{1/n} (K_\rho f(x-t) - f(x-t))\, h_n(t)\,dt \Big\|_{L(a',b')} \leqslant$$

$$\leqslant \| K_\rho f - f \|_{L(a'',b'')} \int_{-1/n}^{1/n} |h_n(t)|\,dt \leqslant \| K_\rho f \|_{L(a,b)} \int_{-1}^{1} h(x)\,dx = 0(\rho^{-\alpha})\ \rho \to \infty$$

where we have put $(a'', b'') = (a' - 1/n,\ b' + 1/n) \subset (a, b)$.

We can now apply Theorem 3.1 to the functions $f_n \in F_{\alpha+1}$ to obtain the existence of $g_n \in BV(a, b)$ such that

$$\int_{-\infty}^{\infty} f_n(x)\,\varphi^{\{\alpha\}}(x)\,dx = \int_{a'}^{b'} \varphi(x)\,dg_n(x), \quad \forall\, \varphi \in \mathcal{D}(a', b') \tag{25}$$

from the proof od that theorem being obvious that

$$\| g_n \|_{BV(a',n')} \leqslant \rho^\alpha \| K_\rho f_n - f_n \|_{L(a',b')} = 0(1), \text{ uniformly in } n. \tag{26}$$

On the other hand, by Lemma 1.3 f_n has an α th Riesz derivative in $F_{\alpha+1}$ $f_n^{\{\alpha\}} = f * h_n^{\{\alpha\}}$. By Lemma 1.4 it follows

$$\int_{-\infty}^{\infty} f_n(x)\,\varphi^{\{\alpha\}}(x)\,dx = \int_{a'}^{b'} \varphi(x) f_n^{\{\alpha\}}(x)\,dx, \quad \forall\, \varphi \in \mathcal{D}(a', b')$$

so that from (25) we obtain

$$\int_{a'}^{b'} \varphi(x)\,dg_n(x) = \int_{a'}^{b'} \varphi(x) f_n^{\{\alpha\}}(x)\,dx, \quad \forall\, \varphi \in \mathcal{D}(a', b')$$

whence

$$dg_n(x) = f_n^{\{\alpha\}}(x)\,dx \text{ a.e. in } (a', b')$$

so that by (26) it follows

$$\| f_n^{\{\alpha\}} \|_{L(a',b')} = 0(1), \text{ uniformly in } n. \tag{27}$$

On the other hand, by Corollary 2.1 we have $\|R_\varepsilon^\alpha f_n\|_{\alpha+1}<C$; $\varepsilon\to 0$. and by (27) the constant C does not depend on n in (a', b'), i.e.

$$\|R_\varepsilon^\alpha f_n\|_{L(a',b')}=O(1),\quad \varepsilon\to 0,\ \text{uniformly in } n. \tag{28}$$

Since $\|f_n-f\|_{\alpha+1}\to 0$, $n\to\infty$ and R_ε^α is a bounded linear operator, letting n tend to infinity in (28) we have

$$\|R_\varepsilon^\alpha f\|_{L(a',b')}=O(1),\quad \varepsilon\to 0$$

which completes the proof of (ii).

(i) From the assumption $\|K_\rho f-f\|_{L(a,b)}=o(\rho^{-\alpha})$, $\rho\to\infty$, we can deduce

$$\|R_\varepsilon^\alpha f_n\|_{L(a',b')}=o(1),\quad \varepsilon\to 0,\ \text{uniformly in } n \tag{29}$$

in the same way (28) was obtained in the proof of part (ii).

Since $f_n=f*h_n$ is smooth, by the definition of the operator R_ε^α we get

$$\lim_{\varepsilon\to 0}\Big\|R_\varepsilon^\alpha f_n-\int_0^\infty \frac{\bar{\Delta}_u^{2m} f_n}{u^{1+\alpha}}\,du\Big\|_{L(a',b')}=0$$

so that by (29) it follows

$$\int_0^\infty \frac{\bar{\Delta}_u^{2m} f_n(x)}{u^{1+\alpha}}\,du=0\quad \text{a.e. } x\in(a',b')$$

whence $\bar{\Delta}_u^{2m} f_n(x)=0$, which in view of the smoothness of f_n implies $f_n^{(2m)}(x)=0$, $x\in(a',b')$. This in turn means that f_n equals a polynomial of degree $2m$ in (a',b'), from which it finally follows that f, being a uniform limit in (a',b') of polynomials, is itself a polynomial of degree $2m$ in (a',b').

Thus the inverse part of the local saturation theorem is proved. The proof of the direct part leans on the fact that the existence of an αth Riesz derivative for f implies the existence of αth Riesz derivative for $h\cdot f$, where $h\in\mathfrak{D}$.

Lemma 3.1. ([9]). *Let $f\in F_{\alpha+1}$ for $\alpha>0$ and $h\in\mathfrak{D}(a,b)$. Then*

$$\|R_\varepsilon^\alpha f\|_{L(a,b)}=O(1),\ \varepsilon\to 0\ \Rightarrow\ \|R_\varepsilon^\alpha(hf)\|_{L^1}=O(1),\ \varepsilon\to 0$$

This Lemma was proved in [9] for the spaces L^p. It is obvious, in view of the properties of the operator R_ε^α (Corollary 2.1) that it is also true in the spaces $F_{\alpha+1}$.

Theorem 3.3. *Let f and k satisfy the assumptions of theorem 3.1. Then*

$$\|R_\varepsilon^\alpha f\|_{L(a,b)}=O(1),\ \varepsilon\to 0\ \Rightarrow\ \rho^\alpha\|K_\rho f-f\|_{L(a',b')}=O(1).$$

Proof. Let $h\in\mathfrak{D}(a,b)$ be such that $h(x)=1$ for $x\in(c,d)\subset(a,b)$ and let $(a',b')\subset(c,d)$.

Let $f\in F_{\alpha+1}$ be such that $\|R_\varepsilon^\alpha f\|_{L(a,b)}=O(1)$, $\varepsilon\to 0$. Then according to Lemma 3.1 it follows $\|R_\varepsilon^\alpha(hf)\|_{L^1}=O(1)$, $\varepsilon\to 0$.

Applying the global theorem (2.1) to the function $h \cdot f$ we obtain

(30) $$\| K_\rho (hf) - (hf) \|_{L(R)} = 0(\rho^{-\alpha}), \qquad \rho \to \infty.$$

Since for $x \in (a', b')$, $h(x) f(x) = f(x)$

$$\| K_\rho f - f \|_{L(a', b')} \leq \| K_\rho f - K_\rho (hf) \|_{L(a', b')} + \| K_\rho (hf) - (hf) \|_{L(a', b')}$$

Thus, in view of (30), the proof will be completed if we show

(31) $$\| K_\rho f - K_\rho (hf) \|_{L(a', b')} = 0(\rho^{-\alpha}), \qquad \rho \to \infty.$$

To this end observe

$$K_\rho (hf)(x) - K_\rho f(x) = \int_{-\infty}^{\infty} (h(t) - 1) f(t)\, k_\rho (x - t)\, dt =$$

$$= \int_{t \notin (c, d)} (h(t) - 1) f(t)\, k_\rho (x - t)\, dt$$

so that it follows

$$\| K_\rho (hf) - K_\rho f \|_{L(a', b')} = \int_{a'}^{b'} \Big| \int_{t \notin (c, d)} (h(t) - 1) f(t)\, k_\rho (x - t)\, dt \Big|\, dx \leq$$

$$\leq 2 \int_{a'}^{b'} \int_{t \notin (c, d)} | f(t)\, k_\rho (x - t) |\, dt\, dx$$

If we introduce the change of variables $x - t = u$ in the last integral, using the fact that $x \in (a', b')$ and $t \notin (c, d) \supset (a', b')$, we get $|u| > \delta = \min(|a' - c|, |b' - d|)$. Hence

$$\| K_\rho (hf) - K_\rho f \|_{L(a', b')} \leq 2 \int_{a'}^{b'} \int_{|u| > \delta} \frac{|f(x - u)|}{|u|^{\alpha+1}} |u|^{\alpha+1} |k_\rho (u)|\, du\, dx \leq$$

$$\leq 2 \sup_{|u| > \delta} |u|^{\alpha+1} |k_\rho (u)| \int_{a'}^{b'} \int_{|u| > \delta} \frac{|f(x - u)|}{|u|^{\alpha+1}}\, du\, dx \leq$$

$$\leq 2 \sup_{|t| > \rho\delta} \left| \frac{t}{\rho} \right|^{\alpha+1} \rho\, |k(t)| \cdot C \|f\|_{\alpha+1}$$

$$\leq C_1 \rho^{-\alpha} \sup_{|t| > \rho\delta} |t|^{\alpha+1} |k(t)| = 0(\rho^{-\alpha}), \qquad \rho \to \infty.$$

Thus (31) is established and this completes the proof of the theorem.

Let us finally remark that the global and local saturation theorem from Sections 2 and 3 can be proved along the same lines for the spaces L^p. It is readily seen that in this case condition (*B*) in Theorems 2.1 is superfluous, while condition (*A*) in Theorems 3.1. – 3 remains unaltered.

We thus conclude that even in spaces L^p a certain smoothness of the function $\hat{k}$ is required for the local saturation, in contrast to the global saturation which depends only on the properties of $\hat{k}$ in the neighbourhood of zero (condition (*S*)).

For example, the operator of Riesz means, whose kernel

$$\hat{r}_{m,n}(v)=\begin{cases}(1-|v|^m)^n, & |v|\le 1\\ 0, & |v|>1\end{cases}$$

obviously satisfies conditions (S) and (M), with $\alpha=m$, is globally saturated for every $m, n>0$. However, if $m>n$, $n\notin N$, the mth derivative of $\hat{r}_{m,n}$ becomes unlimited in the neighbourhood of the point $v=\pm 1$, so that the operator is not locally saturated.

Finally, the following conclusion can be derived from the above. If the singular integral (which satisfies the standard conditions (S) and (M) and is thus globally saturated) is „good enough" to „distinguish" local properties of functions in L^p (i.e. satisfies the conditions $(1+x^2)^{\frac{\alpha+1}{2}}|k(x)|\le C)$, it is also good enough for the approximation of functions from all the spaces F_β, $\beta\le\alpha+1$ and has in all these spaces the same order of saturation.

Remarks. 1) J. Kučera [7] has developed a very general theory of multipliers of temperate distributions. However, his approach does not suit our needs, since the space of temperate functions is decomposed into spaces whose elements f are such that $f(x)\ (1+x^2)^{-\alpha/2}\in L^2$ (instead of $\in L^1$, as in Definition 1.1). We see that these spaces are not appropriate for convolution with k satisfying $(1+x^2)^{\alpha/2}k(x)\in L^\infty$, and since this condition for k is a natural one and indeed satisfied by many important operators, we are bound to introduce the spaces F_α.

2) G. Sunouchi [10] has proved a local saturation theorem for the spaces of periodic functions. However, his method of proof strongly leans on the representation of functions by their Fourier integral $\int_{-\infty}^{\infty}\hat{f}(u)\,e^{iux}\,du$ and obviously cannot be extended to L^p spaces, $p>2$.

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